Ex. 3.9.1. The generating function is $\phi(s) = e^{1.1(s-1)}$, so

$$
\begin{align*}
    u_0 &= 0 \\
    u_1 &= \phi(u_0) = e^{1.1(0-1)} = 0.332871 \\
    u_2 &= \phi(u_1) = e^{1.1(0.000183480-1)} = 0.480061 \\
    u_3 &= \phi(u_2) = e^{1.1(0.332938273-1)} = 0.564433 \\
    u_4 &= \phi(u_3) = e^{1.1(0.480096622-1)} = 0.619326 \\
    u_5 &= \phi(u_4) = e^{1.1(0.564455508-1)} = 0.657874.
\end{align*}
$$

The extinction probability $u_\infty$ is the smaller solution of $u = e^{1.1(u-1)}$. Using EXCEL I found this solution to be 0.8239 (rounded off to four decimal places) after about 100 iterations. Using the Mathematica command FindRoot, the value 0.823866 (accurate to six decimal places) is found for $u_\infty$.

Ex. 3.9.4. Let us write $p_k = P[\xi = k]$, so that $\phi(s) = \sum_{k=0}^{\infty} s^k p_k$. Because $P[Z = 0] = 0$, the generating function of $Z$ is the power series

$$
\begin{align*}
\sum_{k=1}^{\infty} s^k P[Z = k] &= \sum_{k=1}^{\infty} s^k P[\xi = k|\xi > 0] \\
&= \sum_{k=1}^{\infty} s^k P[\xi = k] / P[\xi > 0] \\
&= \frac{\phi(s) - p_0}{1 - p_0},
\end{align*}
$$

for $0 \leq s \leq 1$.

Pr. 3.9.2. Considering only the male descendants in the present branching process, the offspring distribution is as follows:

$$
\begin{align*}
p_0 &= \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32} \\
p_1 &= \frac{1}{4} \cdot \frac{3}{8} + \frac{3}{4} = \frac{27}{32} \\
p_2 &= \frac{1}{4} \cdot \frac{3}{8} = \frac{3}{32} \\
p_3 &= \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}.
\end{align*}
$$
If there are three children, the chances of 0, 1, 2, 3 boys (respectively) are $1/8, 3/8, 3/8,$ and $1/8.$ The corresponding generating function is

$$\phi(s) = \frac{1}{32}(1 + 27s + 3s^2 + s^3).$$

The associated mean number of (male) offspring is

$$\phi'(1) = \frac{27 + 6 + 3}{32} = \frac{36}{32} > 1.$$  

The extinction probability $u_\infty$ is the smaller solution of

$$\frac{1}{32}(1 + 27s + 3s^2 + s^3) = s$$

for $0 \leq s \leq 1$. Clearing out the denominator and moving all terms to one side, the equation becomes

$$s^3 + 3s^2 - 5s + 1 = 0.$$  

But

$$s^3 + 3s^2 - 5s + 1 = (s - 1)(s^2 + 4s - 1).$$

The two roots of the quadratic in the above equation are

$$-2 \pm \sqrt{5},$$

and the only one of these in $[0, 1]$ is

$$-2 + \sqrt{5} = 0.230067977...$$

This is the extinction probability.

**Pr. 3.9.6.** The extinction probability $u_\infty$ is the solution in $(0, 1)$ of the equation

$$u = au^2 + bu + c.$$  

By the quadratic formula,

$$u_\infty = \frac{-(b - 1) - \sqrt{(b - 1)^2 - 4ac}}{2a}.$$  

(We use the minus sign to obtain the smaller solution.) By assumption, the mean value $\mu = b + 2a$ is greater than 1. Let us use this and the fact that $a + b + c = 1$ (so that
1 - b = a + c) to deduce that 1 < (1 - a - c) + 2a = 1 + a - c, which in turn implies that a > c. Using this information we simplify the above expression:

\[ u_\infty = \frac{a + c - \sqrt{(a + c)^2 - 4ac}}{2a} = \frac{a + c - \sqrt{(a - c)^2}}{2a} = \frac{a + c - a - c}{2a} = \frac{c}{a}. \]

**Pr. 3.9.7.** Let \( p_k = P[\xi = k] \). Evidently, \( p_0 = P[GG] = 1/4 \), \( p_1 = P[GB] + P[BG] = 1/2 \), and \( p_k = P[k \text{ boys followed by a girl}] = (1/2)^{k+1} \) for \( k = 2, 3, \ldots \) (Cf. Problem 3.8.3(b).) The corresponding generating function is

\[ \phi(s) = \frac{1}{4} + \frac{s}{2} + \sum_{k=2}^{\infty} \frac{s^k}{2^{k+1}} = \frac{1}{4} + \frac{s}{2} + \frac{s^2/8}{1 - s/2} = \frac{1}{4} + \frac{s}{2} + \frac{s^2}{8 - 4s}. \]

Thus,

\[ \phi'(s) = \frac{1}{2} + \frac{(8 - 4s)2s - s^2(-4)}{(8 - 4s)^2}, \]

so

\[ E[\xi] = \phi'(1) = \frac{1}{2} + \frac{12}{16} = \frac{5}{4}. \]

We can also calculate \( E[\xi] \) more directly using the tail sum formula. For this note that

\[ P[\xi \geq k] = \begin{cases} 3/4, & k = 1; \\ (1/2)^k, & k \geq 2. \end{cases} \]

Therefore,

\[ E[\xi] = \sum_{k=1}^{\infty} P[\xi \geq k] = \frac{3}{4} + \sum_{k=2}^{\infty} (1/2)^k = \frac{3}{4} + \frac{(1/2)^2}{1 - 1/2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}. \]

**Ex. 4.1.2.** This transition matrix is clearly regular, so to find the limiting distribution we have only to find the stationary distribution \( \pi \). The matrix equation \( \pi P = \pi \) written as a three-by-three system, becomes

\[ .6\pi_0 + .3\pi_1 + .4\pi_2 = \pi_0 \\
.3\pi_0 + .3\pi_1 + .1\pi_2 = \pi_1 \\
.1\pi_0 + .4\pi_1 + .5\pi_2 = \pi_2 \]
Moving everything to the left side we get
\[-.4\pi_0 + .3\pi_1 + .4\pi_2 = 0\]
\[.3\pi_0 - .7\pi_1 + .1\pi_2 = 0\]
\[.1\pi_0 + .4\pi_1 - .5\pi_2 = 0\]

Solve the last equation for $\pi_2$:
\[\pi_2 = .2\pi_0 + .8\pi_1,\]
and then use this to eliminate $\pi_2$ from the first equation:
\[-.32\pi_0 + .62\pi_1 = 0.\]

That is, $\pi_1 = \frac{16}{31}\pi_0$. Plugging this into the earlier equation for $\pi_2$ we get
\[\pi_2 = .2\pi_0 + .8\pi_1 = .2\pi_0 + \frac{64}{155}\pi_0 = \frac{95}{155}\pi_0.\]

Thus,
\[\pi = [\pi_0, (16/31)\pi_0, (19/31)\pi_0],\]
and for these to add up to 1 we must have $\pi_0 = 31/66$. Thus
\[\pi = [31/66, 16/66, 19/66].\]

**Ex. 4.1.3.** The stationary distribution for this Markov chain is
\[\pi = [\frac{3}{13}, \frac{3}{13}, \frac{7}{13}].\]

Therefore, the long-run fraction of time spent in state 1 is $\pi_1 = 3/13$.

**Ex. 4.1.4.** The stationary distribution for this Markov chain is
\[\pi = [\frac{55}{132}, \frac{24}{132}, \frac{53}{132}].\]

(Proceed as in Exercise 4.1.2.) Because $P$ is regular, the limit transition probabilities coincide with the stationary distribution;
\[\lim_{n} P_{ij}^{(n)} = \pi_j, \quad j = 0, 1, 2.\]
As such, $\pi_j$ represents the long run fraction of time spent in state $j$, so the long run cost per period is
\[
2\pi_0 + 5\pi_1 + 3\pi_2 = \frac{2 \cdot 55 + 5 \cdot 24 + 3 \cdot 53}{132} = \frac{389}{132} = 2.95...
\]

**Ex. 4.1.5.** This transition matrix is regular (by the Simple Sufficient Condition discussed in class), so to find the limiting distribution we have only to find the stationary distribution $\pi$. This amounts to solving the linear system
\[
\begin{align*}
\frac{1}{10}\pi_0 + \pi_3 &= \pi_0 \\
\frac{1}{2}\pi_0 &= \pi_1 \\
\pi_1 &= \pi_2 \\
\frac{4}{10}\pi_0 + \pi_2 &= \pi_3
\end{align*}
\]
subject to the constraint $\sum_{j=0}^3 \pi_j = 1$. By the first of these equation we have $\pi_3 = (9/10)\pi_0$, and then using the second and third equations we deduce that $\pi$ has the form
\[
[ \pi_0 \quad \frac{1}{2}\pi_0 \quad \frac{1}{2}\pi_0 \quad \frac{9}{10}\pi_0 ].
\]
The constraint forces $2.9\pi_0 = 1$, and so $\pi_0 = 10/29$, and
\[
\pi = [10/29 \quad 5/29 \quad 5/29 \quad 9/29].
\]

**Pr. 4.1.5.** With the states listed in alphabetical order, the transition matrix for this Markov chain is
\[
P = \begin{bmatrix}
0 & 1/2 & 0 & 1/2 \\
1/3 & 0 & 1/3 & 1/3 \\
0 & 1 & 0 & 0 \\
1/2 & 1/2 & 0 & 0
\end{bmatrix}.
\]
The stationary distribution is
\[
\pi = [\frac{2}{8} \quad \frac{2}{8} \quad \frac{1}{8} \quad \frac{2}{8}].
\]
Therefore, the long-run probability of finding the train in town $D$ is $1/4$.

**Pr. 4.1.6.** (a) $\pi_j$.
(b) $\pi_k \cdot P_{k,j}$.
(c) $\pi_k \cdot P_{k,j}$. 

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Pr. 4.1.7. By the Simple Sufficient Condition described in class, this transition matrix is regular, so the limit transition probabilities coincide with the stationary distribution. The system $\pi P = \pi$ written out in detail is

\[
\begin{align*}
0.5\pi_0 + \pi_1 &= \pi_0 \\
0.5\pi_2 &= \pi_1 \\
(1/3)\pi_2 + \pi_3 &= \pi_2 \\
0.5\pi_0 + (1/6)\pi_2 &= \pi_3
\end{align*}
\]

By the first equation, $\pi_1 = 0.5\pi_0$, and then using the second equation we see that, $\pi_2 = \pi_0$. Finally, by the third equation, $\pi_3 = (2/3)\pi_0$. So that these add up to 1 we must have $\pi_0 = 15/67$. Thus, $\pi = [6/19 \ 3/19 \ 6/19 \ 4/19]$

Pr. 4.1.10. If the number of states, $N + 1$, is composite, then $P$ need not be regular. For example, if $N + 1$ factors as $ab$ ($a$ and $b$ integers greater than 1) and if $p_a > 0$ but $p_j = 0$ for all $j \in \{1, 2, \ldots, N\}$ except $a$, then $P_{01}^{(n)} = 0$ for all $n$. Indeed, the state space $S$ comprises $b$ communicating classes, each with $a$ elements.

In general, for $N + 1$ composite, the limiting transition probability $\lim_n P_{ij}^{(n)}$ exists, but is non-zero only if $i$ and $j$ are in the same class. For such $i$ and $j$, the limit probability is the uniform distribution on the class to which $i$ and $j$ belong. The structure of the communicating classes depends on which of the $p_0, p_1, \ldots, p_N$ are equal to 0.

If $N + 1$ is prime, then $P$ is regular, and the limit distribution coincides with the (unique) stationary distribution, which is the uniform distribution over $\{0, 1, \ldots, N\}$. To see this observe that since $0 < p_0 < 1$, we must have $p_k > 0$ for some $k \in \{1, 2, \ldots, N\}$. Let $b = N + 1$, which we are assuming is a prime. Since $b$ is a prime, the ring of integers mod $b$, $\mathbb{Z}_b$, is a field. Thus for each pair $i$ and $j$ of distinct elements of the state space there exists $n \in \{1, 2, \ldots, N\}$ (depending on $i$ and $j$) such that $nk = (j - i)$ (modulo $b$). We then have $P_{ij}^{(n)} \geq p_k^n > 0$. Because $P_{ii} = p_0 > 0$ for all $i$, it now follows from the Simple Sufficient Condition discussed in class that $P$ is regular.