Math 180B, Winter 2019
Homework 9 Solutions

Ex. 5.2.1. (a) \( P[X = 0] = (0.94)^20 = 0.2901; \ P[X = 1] = 20(0.06)(0.94)^{19} = 0.3703; \ P[X = 2] = 190(0.06)^2(0.94)^{18} = 0.2246. \)

(b) \( P[X = 0] = (0.97)^{40} = 0.2957; \ P[X = 1] = 40(0.03)(0.97)^{39} = 0.3658; \ P[X = 2] = 780(0.03)^2(0.97)^{38} = 0.2206. \)

(c) \( P[X = 0] = e^{-1.2} = 0.3012; \ P[X = 1] = e^{-1.2}(1.2) = 0.3614; \ P[X = 2] = e^{-1.2}(1.2)^2/2 = 0.2169. \)

Pr. 5.2.4. The exact distribution of the number of points that fall in \([0,1)\) is binomial with parameters \(N\) and \(p = 1/N\). Because \(Np = 1\), the limit distribution, as \(N \to \infty\), is the Poisson distribution with \(\lambda = 1\).

Pr. 5.2.5. The exact distribution of the number of points that fall within one unit of the origin is binomial with parameters \(N\) and \(p = (\pi \cdot 1^2)/(\pi r^2) = r^{-2}\). Because \(Np = Nr^{-2} = \lambda\pi\), the limit distribution, as \(N \to \infty\), is the Poisson distribution with parameter \(\lambda\pi\).

Pr. 5.2.8. We have

\[
\begin{align*}
P[S_n = 0] & = 0.3024, & P[X(1) = 0] & = 0.3679 \\
P[S_n = 1] & = 0.4404, & P[X(1) = 1] & = 0.3679 \\
P[S_n = 2] & = 0.2144, & P[X(1) = 2] & = 0.1839 \\
P[S_n = 3] & = 0.0404, & P[X(1) = 3] & = 0.0613 \\
P[S_n = 4] & = 0.0024, & P[X(1) = 4] & = 0.0153.
\end{align*}
\]

These are in accord with the error bound (5.7) (page 233) because \(\sum_{i=1}^{4} p_i^2 = 0.3\). Indeed, \[
\begin{align*}
|P[S_n = 0] - P[X(1) = 0]| & = 0.0655 \\
|P[S_n = 1] - P[X(1) = 1]| & = 0.0725 \\
|P[S_n = 2] - P[X(1) = 2]| & = 0.0305 \\
|P[S_n = 3] - P[X(1) = 3]| & = 0.0209 \\
|P[S_n = 4] - P[X(1) = 4]| & = 0.0129.
\end{align*}
\]
Ex. 5.3.8.

\[
P[X(1) = 5|X(2) = 12] = \frac{P[X(1) = 5, X(2) = 7]}{P[X(2) = 12]} = \frac{e^{-\frac{5^2}{5!}} \cdot e^{-\frac{5^7}{7!}}}{e^{-\frac{10^2}{12^2}} / 12!} = \left(\frac{12}{5}\right)^{(1/2)^{12}} = 0.1934.
\]

Pr. 5.3.9. Let \(W_k^{(j)}\) be the arrival time of the \(k\)th “mark” in Poisson process number \(j\) \((j = 1, 2)\).

(a) Because \(W_1^{(j)}\) has the exponential distribution with parameter \(\lambda_j\), this probability is

\[
P[W_1^{(1)} < W_1^{(2)}] = \int_0^\infty \int_0^\infty \lambda_1 \lambda_2 e^{-\lambda_1 s - \lambda_2 t} dt ds
= \int_0^\infty \lambda_1 e^{-(\lambda_1 + \lambda_2) s} ds
= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]

(b) Because \(W_2^{(j)}\) has the gamma distribution with parameters \((2, \lambda_j)\), this probability is

\[
P[W_2^{(1)} < W_2^{(2)}] = \int_0^\infty \int_0^\infty \lambda_1^2 \lambda_2^2 s e^{-\lambda_1 s - \lambda_2 t} dt ds
\]

The inner integral can be integrated by parts:

\[
\int_s^\infty te^{-\lambda_2 t} dt = -\left(\frac{t}{\lambda_2} + \frac{1}{\lambda_2^2}\right) e^{-\lambda_2 t} \bigg|_s^\infty = \left(\frac{s}{\lambda_2} + \frac{1}{\lambda_2^2}\right) e^{-\lambda_2 s}.
\]

Therefore,

\[
P[W_2^{(1)} < W_2^{(2)}] = \int_0^\infty \lambda_1^2 \lambda_2^2 (s^2 e^{-(\lambda_1 + \lambda_2) s} + \frac{s}{\lambda_2} e^{-(\lambda_1 + \lambda_2) s}) ds
= \lambda_1^2 \lambda_2 \int_0^\infty s^2 e^{-(\lambda_1 + \lambda_2) s} ds + \lambda_1^2 \int_0^\infty se^{-(\lambda_1 + \lambda_2) s} ds
= \frac{2\lambda_1^2 \lambda_2}{(\lambda_1 + \lambda_2)^3} + \frac{\lambda_1^2}{(\lambda_1 + \lambda_2)^2}
= \frac{2\lambda_1^2 \lambda_2 + \lambda_1^3 + \lambda_1^2 \lambda_2}{(\lambda_1 + \lambda_2)^3}
= \frac{\lambda_1^3 + 3\lambda_1^2 \lambda_2}{(\lambda_1 + \lambda_2)^3}.
\]

Second Solution. Let \(X = \{X(t) : t \geq 0\}\) be a Poisson process with rate \(\lambda_1 + \lambda_2\). Let \(p = \lambda_1/(\lambda_1 + \lambda_2)\) and \(q = 1 - p = \lambda_2/(\lambda_1 + \lambda_2)\). Using a device to be discussed in class, color
the points of $X$ at random, either Red (with probability $p$) or Green (with probability $q$),
the different points being colored independently of each other. Then the Red points form
a Poisson process $X_1$ of rate $p \cdot (\lambda_1 + \lambda_2) = \lambda_1$, the Green points form a Poisson process
$X_2$ of rate $q \cdot (\lambda_1 + \lambda_2) = \lambda_2$, and $X_1$ and $X_2$ are independent.

(a) Thought of in terms of the coloring construction just discussed, the required prob-
bility is just the probability that the first point of $X$ is colored Red; this probability is
$p = \lambda_1/(\lambda_1 + \lambda_2)$.

(b) In terms of the coloring construction just discussed, we now seek the probability
that as the points are colored, two are colored Red before two are colored Green. This
can happen in one of three ways: the points are colored RR or RGR or GRR. These three
events have respective probabilities $p^2$, $pqp = p^2q$, and $qp^2$. The required probability is
therefore

$$p^2 + 2p^2q = \frac{\lambda_1^2}{(\lambda_1 + \lambda_2)^2} + \frac{2\lambda_1^2\lambda_2}{(\lambda_1 + \lambda_2)^3}$$
$$= \frac{\lambda_1^2(\lambda_1 + \lambda_2) + 2\lambda_1^2\lambda_2}{(\lambda_1 + \lambda_2)^3}$$
$$= \frac{\lambda_1^3 + 3\lambda_1^2\lambda_2}{(\lambda_1 + \lambda_2)^3}.$$