1. [20 points]. On my desk I keep a stack of three books. The books are numbered 1, 2, and 3. Each morning I pick a book at random from the stack, read one page, and then place the book on the top of the stack. Book number \( i \) is picked with probability \( p_i \). [Assume \( 0 < p_i < 1 \) and \( p_1 + p_2 + p_3 = 1 \).] The arrangement of the stack of books changes randomly from day to day; in fact it evolves according to a Markov chain. The state space of this Markov chain is the set of permutations of \( \{1,2,3\} \), namely \( \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\} \). For example, the state \( (3,1,2) \) corresponds to the stack in which book 3 is on top, book 1 is in the middle, and book 2 is on the bottom. If I were to choose book 1 from this stack, then after the book was returned the new state would be \( (1,3,2) \). Write down the (six-by-six) transition matrix for this Markov chain.

Solution. With the states listed in the indicated order, we have

\[
P = \begin{bmatrix}
p_1 & 0 & p_2 & 0 & p_3 & 0 \\
0 & p_1 & p_2 & 0 & p_3 & 0 \\
p_1 & 0 & p_2 & 0 & 0 & p_3 \\
p_1 & 0 & 0 & p_2 & 0 & p_3 \\
0 & p_1 & 0 & p_2 & p_3 & 0 \\
0 & p_1 & 0 & p_2 & 0 & p_3 \\
\end{bmatrix}.
\]

For example, if we are in state \( (2,1,3) \) a transition to \( (1,2,3) \) occurs (with probability \( p_1 \)) if book 1 is chosen, a transition to \( (2,1,3) \) (that is, no change) occurs (with probability \( p_2 \)) if book 2 is chosen, a transition to \( (3,2,1) \) occurs (with probability \( p_3 \)) if book 3 is chosen.

2. [20 points]. A Markov chain \( \{X_n : n = 0, 1, 2, \ldots\} \) has state space \( \{0,1,2\} \) and transition matrix

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 \\
0 & 1/2 & 1/2 \\
1/2 & 0 & 1/2 \\
\end{bmatrix}.
\]

(a) Explain briefly why \( P \) is regular.

(b) Find \( \lim_{n \to \infty} P_{0j}^{(n)} \) for \( j = 0, 1, 2 \).

Solution. (a) We can apply the Simple Sufficient Condition discussed in class, because \( P_{22} > 0 \) and because this transition matrix is clearly irreducible—it is possible to move from any one state to any other state in 3 (or more) steps.

(b) First, we seek the stationary distribution by solving \( \pi = \pi P \). Written as a system
of three equations this becomes

\[ \begin{align*}
\pi_0 &= (1/2)\pi_2, \\
\pi_1 &= (1/2)\pi_0 + (1/2)\pi_1, \\
\pi_2 &= (1/2)\pi_0 + (1/2)\pi_1 + (1/2)\pi_2.
\end{align*} \]

The first of these equations implies that \( \pi_2 = 2\pi_0 \); solving the second for \( \pi_0 \) we see that \( \pi_0 = \pi_1 \). Thus,

\[ \pi = [\pi_0 \quad \pi_0 \quad 2\pi_0]. \]

In order that the sum of the coordinates of \( \pi \) is 1, we must have \( \pi_0 = 1/4 \), so

\[ \pi = [1/4 \quad 1/4 \quad 1/2]. \]

Because \( \mathbf{P} \) is regular, the limit theorem applies, and we deduce that

\[ \lim_{n \to \infty} P^{(n)}_{ij} = \pi_j. \]

In particular,

\[ \begin{align*}
\lim_{n \to \infty} P^{(n)}_{00} &= 1/4, \\
\lim_{n \to \infty} P^{(n)}_{01} &= 1/4, \\
\lim_{n \to \infty} P^{(n)}_{02} &= 1/2.
\end{align*} \]

3. [20 points]. Let \( \{X_n : n = 0, 1, 2, \ldots\} \) be a branching process with \( X_0 = 1 \), and with offspring distribution whose generating function is

\[ \phi(s) = (1 + 2s + 2s^3)/5, \quad 0 \leq s \leq 1. \]

(a) Compute \( \mu = E[X_1] \).
(b) Find \( E[X_n] \).
(c) Find the extinction probability \( u_\infty \).

Solution. (a) We have \( \phi'(s) = (2 + 6s^2)/5 \) so that \( \mu = E[X_1] = \phi'(1) = 8/5 \).

Alternatively, from the form of the generating function \( \phi \) we see that \( p_0 = 1/5, p_1 = 2/5, p_3 = 2/5 \), and so

\[ \mu = E[X_1] = E[\xi] = 0 \cdot p_0 + 1 \cdot p_1 + 3 \cdot p_3 = 1 \cdot (2/5) + 3 \cdot (2/5) = 8/5. \]

(b) We know that for a branching process with \( X_0 = 1 \) we have \( E[X_n] = \mu^n \). Thus, \( E[X_n] = (8/5)^n \) for \( n = 0, 1, 2, \ldots \).
(c) Because $\mu > 1$, we know that the equation $\phi(s) = s$ has two solutions in $[0, 1]$, and that $u_\infty$ is the smaller of the two solutions (the other being $s = 1$). The equation $\phi(s) = s$ written explicitly is
\[
\frac{1 + 2s + 2s^3}{5} = s.
\]
Clearing out the denominator and moving the resulting $5s$ to the left side of the equation, this becomes
\[
2s^3 - 3s + 1 = 0.
\]
But knowing that one of the roots of the cubic equation $2s^3 - 3s + 1$ is 1 allows us to factor (using synthetic division, for example) to obtain
\[
2s^3 - 3s + 1 = (s - 1)(2s^2 + 2s - 1).
\]
By the quadratic formula, the second factor above has roots
\[
-1 \pm \sqrt{3} \over 2,
\]
and the one of these that is in $(0, 1)$ is $(-1 + \sqrt{3})/2$. Thus
\[
u_\infty = (\sqrt{3} - 1)/2 = 0.366025...\]

4. [20 points]. Consider the Markov chain with state space $\{0, 1, 2\}$ and transition matrix
\[
P = \begin{bmatrix}
1/3 & 1/3 & 1/3 \\
1/2 & 0 & 1/2 \\
0 & 0 & 1
\end{bmatrix}.
\]
Use first-step analysis to compute the mean time to reach the absorbing state 2, starting from state 0.

Solution. By the matrix method for first-step analysis, we have
\[
Q = \begin{bmatrix}
1/3 & 1/3 \\
1/2 & 0
\end{bmatrix},
\]
and then
\[
I - Q = \begin{bmatrix}
2/3 & -1/3 \\
-1/2 & 1
\end{bmatrix},
\]
which has determinant $1/2$. Therefore
\[
(I - Q)^{-1} = \begin{bmatrix}
2 & 2/3 \\
1 & 4/3
\end{bmatrix}.
\]
Finally, \( v_0 = E[T_2 | X_0 = 0] \) is the sum of the elements in row 0 of this last matrix; namely, 
\[ v_0 = 2 + 2/3 = 8/3 = 2.666... \]

Alternatively, we can slog through solving the system of two equations for \( v_i = E[T_2 | X_0 = i] \): By first-step analysis,
\[
\begin{align*}
v_0 &= 1 + (1/3)v_0 + (1/3)v_1, \\
v_1 &= 1 + (1/2)v_0.
\end{align*}
\]

Using the second equation to eliminate \( v_1 \) from the first we obtain
\[
v_0 = 1 + (1/3)v_0 + (1/3)[1 + (1/2)v_0] = 4/3 + (1/2)v_0,
\]
which yields \( v_0 = 8/3 \) as before.

5. [20 points]. Consider a Markov chain \( \{X_n : n = 0, 1, 2, \ldots\} \) with state space \( \{0, 1, 2, 3\} \) and transition matrix
\[
P = \begin{bmatrix}
.5 & 0 & .5 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & .1 & .9 \\
0 & 0 & .9 & .1
\end{bmatrix}.
\]

(a) Find the communicating classes for this Markov chain.

(b) Is this Markov chain irreducible? (Explain.)

(c) Which states are transient? Which are recurrent?

Solution. It’s helpful to make a “transitions diagram” to see what’s going on.

(a) The chain lingers in state 0 for a while, then moves on to state 2, never to return to 0. Thus \( \{0\} \) is one communicating class, call it \( C_1 \). State 1, being an absorbing state, is in a class by itself, say \( C_2 = \{1\} \). Finally, the chain moves freely between states 2 and 3, but never leaves \( \{2, 3\} \) once there, thus there is a third communicating class \( C_3 = \{2, 3\} \).

(b) Because there is more than one communicating class, the Markov chain is not irreducible.

(c) State 0 is transient, per the discussion for part (a), and state 1, being absorbing, is recurrent. Finally, both 2 and 3 are recurrent as well.