2. Consider a two-period \((T = 2)\) binomial model with initial stock price \(S_0 = 8\), \(u = 2\), \(d = 1/2\), and “real world” up probability \(p = 1/3\).

(a) Draw the binary tree illustrating the possible paths followed by the stock price process.

(b) The sample space for this problem can be listed as \(\Omega = \{dd, du, ud, uu\}\). List the probabilities associated with the individual elements of the sample space \(\Omega\).

(c) List the events (i.e., the subsets of \(\Omega\)) making up the \(\sigma\)-field \(\mathcal{F}_1\) determined by \(S_1\).

(d) List the values (one for each path) of a European contingent claim whose payoff at \(T = 2\) is \(X = S_0 + S_1 + S_2\).

Solution. (a)

\[
\begin{align*}
S_0 &= 8 \\
S_1^d &= 4 & S_1^u &= 16 \\
S_2^{dd} &= 2 & S_2^{du} &= 8 & S_2^{ud} &= 8 & S_2^{uu} &= 32
\end{align*}
\]

(b) The \(P\) probabilities of the individual elements of \(\Omega\) are (in the order in which those points are listed above):

\[
\frac{4}{9} \quad \frac{2}{9} \quad \frac{2}{9} \quad \frac{1}{9}.
\]

(c) \(\mathcal{F}_1 = \{\emptyset, \Omega, \{dd, du\}, \{ud, uu\}\}\).

(d) In the same order as above, the values taken by \(X\) are

\[
14 \quad 20 \quad 32 \quad 56.
\]

3. Consider a single-period binomial model with \(r = 1/3\), \(S_0 = 2\), \(d = 5/4\), \(u = 3/2\). Let us take \(\Omega = \{\omega_1, \omega_2\}\) and \(S_1(\omega_1) = 5/2\) and \(S_1(\omega_2) = 3\).

(a) For the trading strategy \(\phi = (\alpha, \beta) = (3, -2)\), compute \(V_0(\phi)\), \(V_1(\phi)(\omega_1)\), and \(V_1(\phi)(\omega_2)\). [Recall that \(\alpha\) is the holdings in the stock and \(\beta\) the holdings in the bond.]

b) Let \(X\) be a European call option with strike price \$2.50 and expiration time \(T = 1\).

(i) Find \(X(\omega_1)\) and \(X(\omega_2)\).
(ii) Find the replicating strategy $\phi$ for $X$
(iii) Find the manufacturing cost $V_0(\phi)$ for that strategy.

**Solution.**
(a) $V_0(\phi) = 3 \cdot S_0 - 2 = 4$. $V_1(\phi) = 3S_1 - 8/3$, so $V_1(\phi)(\omega_1) = 3(5/2) - 2(4/3) = 29/6$ and $V_1(\phi)(\omega_2) = 3 \cdot 3 - 2(4/3) = 19/3$.

(b) $X = (S_1 - 2.5)^+$.
(i) $X(\omega_1) = 0$, $X(\omega_2) = .5$.
(ii) The replicating strategy for $X$ is given by the familiar formulae:

$$\alpha_1 = \frac{X(\omega_2) - X(\omega_1)}{u-d} \cdot \frac{1}{S_0} = \frac{.5 - 0}{3/2 - 5/4} \cdot \frac{1}{2} = 1,$$

while

$$\beta_1 = \frac{uX(\omega_1) - dX(\omega_2)}{u-d} \cdot \frac{1}{B_1} = \frac{3/2 \cdot 0 - 5/4 \cdot .5}{3/2 - 5/4} = -\frac{15}{8}.$$  

(c) The cost of the replicating strategy $(1, -15/8)$ found in part (b) is $V_0 = 1 \cdot 2 - 15/8 \cdot 1 = 1/8$.

4. This problem is about a multi-period binomial model model with $T = 2$, $B_0 = 1$, $S_0 = $100, $S_1 = $200 or $50$, and $r = 0.1$. In this case $p^* = 2/5$. Now consider an American put option with strike price $K = $120.

(a) Use backward recursion in binary tree for the necessary wealth process $U_t$, based on the payoffs $Y_t = (120 - S_t)^+$, to compute the arbitrage-free initial price of the American put option.

(b) Suppose that you can buy the American put option at time zero for $1 less than its arbitrage-free price. Explicitly describe a strategy that yields an arbitrage opportunity for a buyer of the American put option.

**Solution.**
(a) $S_0 = 100$

$S_1^d = 50 \quad S_1^u = 200$

$S_2^{dd} = 25 \quad S_2^{du} = 100 \quad S_2^{ud} = 100 \quad S_2^{uu} = 400$

Also, writing $Y_t = (120 - S_t)^+$ for the put-option payoff at time $t$,

$Y_0 = 20$

$Y_1^d = 70 \quad Y_1^u = 0$
\[ Y_2^{dd} = 95 \quad Y_2^{du} = 20 \quad Y_2^{ud} = 20 \quad Y_2^{uu} = 0 \]

Using the backward recursion \( U_{t-1} = \max\{Y_{t-1}, (1 + r)^{-1}E^*[U_t | F_{t-1}]\} \) for \( t = 2, 1 \) we obtain the “U tree”:

\[
\begin{align*}
U_0 &= 5100/121 \\
U_1^d &= 70 \quad U_1^u = 120/11 \\
U_2^d &= 95 \quad U_2^{du} = 20 \quad U_2^{ud} = 20 \quad U_2^{uu} = 0
\end{align*}
\]

The no-arbitrage price of this American put option is therefore \( 5100/121 = 42.15 \) dollars.

(b) In view of the U tree of part (a), the optimal time for a holder of the put to exercise the option is the stopping time consisting of “stop at time 1 if the stock price goes down, and stop at time 2 if it goes up”. A buyer’s arbitrage is available if the put is priced at $41.15: Buy the put for $41.15, sell the super-replicating strategy \( \phi^* = (\alpha^*_t, \beta^*_t) \) (which is such that \( V_0(\phi^*) = 42.15 \) and \( V_t(\phi^*) \geq U_t \) for \( t = 0, 1, 2 \)), and put the $1 left over into the bond. If the stock goes down at time 1, exercise the put option, getting $70 thereby. The total worth of the buyer’s portfolio is then \( 70 + V_\tau(\phi^*) = 70 - 63.64 = 6.36 \) (dollars), which could be invested in the bond. If the stock goes up at time 1, exercise the put at time 2; the proceeds then exactly balance the (negative) worth of the superhedging portfolio. This strategy has no downside risk, and there is a guaranteed profit of $\$(11/10)^2 = 1.21\$ (more if the stock goes down at time 1) no matter what happens.

5. Consider the two-period binomial model with \( S_0 = 100, u = 2, d = 1/2, \) and \( r = 1/2. \) As usual, \( \Omega = \{dd, du, ud, uu\}, F_0 = \{\emptyset, \Omega\}, F_1 = \{\{dd, du\}, \{ud, uu\}, \emptyset, \Omega\}, \) and \( F_2 \) is the \( \sigma \)-field of all subsets of \( \Omega. \)

(a) Compute the risk-neutral “up” probability \( p^*. \)

(b) The sequence \( \{M_0, M_1, M_2\} \) is a martingale with respect to the filtration \( \{F_0, F_1, F_2\} \) and the martingale measure \( \mathbf{P}^* \) determined by \( p^* \) from part (a). (That is, \( E^*[M_t | F_{t-1}] = M_{t-1} \) for \( t = 1, 2. \) ) You are told that

\[
M_2(dd) = 2, \quad M_2(du) = 3, \quad M_2(ud) = 7, \quad M_2(uu) = 9.
\]

Compute \( M_1^d, M_1^u, \) and \( M_0. \)

Solution. (a) \( p^* = (1 + r - d)/(u - d) = (1.5 - .5)/(2 - .5) = 2/3. \)

(b)

\[
\begin{align*}
M_1^d &= (1/3)2 + (2/3)3 = 8/3, \quad M_1^u = (1/3)7 + (2/3)9 = 25/3, \\
M_0 &= (1/3)M_1^d + (2/3)M_1^u = (1/3)(8/3) + (2/3)(25/3) = 58/9.
\end{align*}
\]
6. Consider the sample space $\Omega = \{1, 2, 3\}$ and the probability measure $\mathbf{P}$ giving each point weight 1/3. On this probability space consider the finite model with $T = 1$, $S^0_0 = S^0_1 = 1$, $S^1_0 = 2$, and

$$S^1_1(1) = 1, \quad S^1_1(2) = 2, \quad S^1_1(3) = 4.$$ 

(a) This market is viable, so there is at least one EMM. Find all possible EMMs.
(b) Consider the contingent claim $Y$ given by

$$Y(1) = 2, \quad Y(2) = 4, \quad Y(3) = 8.$$ 

Without actually finding a hedging strategy, explain why $Y$ can be replicated.

Solution. (a) In the present context, an EMM is a system of three weights $a, b, c$ (the respective probabilities of 1, 2, 3) such that $a + b + c = 1$, $a > 0$, $b > 0$, $c > 0$, and $a + 2b + 4c = 2$. (The latter is just the condition that the mean of $S^1_1$ be equal to $S^1_0$.) This system of equations is easily solved: subtract the first equation from the second, and then isolate $b$ to find that $b = 1 - 3c$, and then that $a = 1 - b - c = 2c$. The constraints $a > 0$, $b > 0$ force $0 < c < 1/3$. Thus, a complete description of all EMMs is:

$$\mathbf{P}^c \text{ gives weights } 2c, 1 - 3c, c \text{ to } 1, 2, 3 \quad (0 < c < 1/3).$$

(b) We compute:

$$E^c[Y] = c \cdot 2 + (1 - 3c) \cdot 4 + c \cdot 8 = 4$$ 

independently of $c \in (0, 1/3)$. That is, the mean of $Y$ is the same for all EMMs. By the “Duality Formula” discussed in class, this means that $V_+(Y) = V_-(Y)$, so that $Y$ can be replicated (see Corollary 3.5.2, page 49 of the text).

7. This is a continuation of Problem 6: The sample space is $\Omega = \{1, 2, 3\}$ with the probability measure $\mathbf{P}$ giving each point weight 1/3. We have $T = 1$, $S^0_0 = S^0_1 = 1$, $S^1_0 = 2$, and

$$S^1_1(1) = 1, \quad S^1_1(2) = 2, \quad S^1_1(3) = 4.$$ 

(a) Compute $V_+(X)$ and $V_-(X)$ for the contingent claim $X$ given by

$$X(1) = 1, \quad X(2) = 3, \quad X(3) = 9.$$ 

(b) Find a trading strategy $\phi = (\phi^0_1, \phi^1_1)$ such that $V_1(\phi) \geq X$ and $V_0(\phi) = V_+(X)$.

Solution. (a) Using the EMMs $\mathbf{P}^c$ from Problem 6,

$$E^c[X] = 2c \cdot 1 + (1 - 3c) \cdot 3 + c \cdot 9 = 3 + 2c, \quad 0 < c < 1/3.$$
From this and the Duality Formula it follows that

\[ V_+(X) = \sup\{ E^c[X] : 0 < c < 1/3 \} = 11/3 = \frac{11}{3}, \]

and

\[ V_-(X) = \inf\{ E^c[X] : 0 < c < 1/3 \} = 3. \]

Alternatively, to find \( V_+(X) \), we can solve the optimization problem

\[
\begin{align*}
\text{minimize} & \quad 2\alpha + \beta \\
\text{subject to} & \quad \alpha + \beta \geq 1, 2\alpha + \beta \geq 3, 4\alpha + \beta \geq 9.
\end{align*}
\]

(Here I write \( \alpha = \phi_1^1 \) and \( \beta = \phi_0^0 \).) A picture shows that the second constraint is redundant, and then that the minimum is attained at the point \( (\alpha, \beta) = (8/3, -5/3) \) where the boundary lines of the other two constraints intersect. In particular, we obtain \( V_+(X) = V_0(8/3, -5/3) = (8/3)2 - 5/3 = 11/3, \) as before.

Similarly, to find \( V_-(X) \), we can solve the optimization problem

\[
\begin{align*}
\text{maximize} & \quad 2\alpha + \beta \\
\text{subject to} & \quad \alpha + \beta \leq 1, 2\alpha + \beta \leq 3, 4\alpha + \beta \leq 9.
\end{align*}
\]

(Again, I write \( \alpha = \phi_1^1 \) and \( \beta = \phi_0^0 \).) A picture now shows that the maximum must occur at one (or both) of the vertices of the boundary of the constraint region. These two points \((3, -3)\) and \((2, -1)\) are the points where the second and third boundary lines intersect, and where the first and second boundary lines intersect. Since \( V_0(3, -3) = V_0(2, -1) = 3 \), we have \( V_-(X) = 3 \), as expected. Either of these two points represents a maximal subhedging strategy.

(b) [Warning: I will now use the notation \( \phi = (\phi_0^0, \phi_1^1) \), which would be \((\beta, \alpha)\) in the lazy notation used in solving part (a).] In computing \( V_+(X) \) in part (a), by the constrained optimization method, we found a strategy \( \phi = (\phi_0^0, \phi_1^1) = (-5/3, 8/3) \) with \( V_0(\phi) = 11/3 \) and \( V_1(\phi) \geq X \):

\[
V_1(-5/3, 8/3) = (-5/3)S_1^1 + 8/3S_0^0 = \begin{pmatrix} 1 \\ 11/3 \\ 9 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}.
\]