1. Consider a single-period binomial model with $r = 1/5$, $S_0 = 3$, $u = 2$, $d = 1/2$, and $p = 5/7$. Let $X$ be a European put option with strike price $K = $3, expiring at time $T = 1$. Compute the arbitrage free price of this option.

Solution. In the present situation, $p^* = (1.2 - .5)/(2 - .5) = 7/15$. Also, $1/(1 + r) = 5/6$. Since $X^u = (3 - 6)^+ = 0$ and $X^d = (3 - 1.5)^+ = 1.5$, we have

$$V_0 = \frac{5}{6} \cdot \left[ \frac{7}{15} \cdot 0 + \frac{8}{15} \cdot \frac{3}{2} \right] = \frac{2}{3}.$$  

2. Consider a three period ($T = 3$) binomial model with initial stock price $S_0 = $8, $u = 3$, $d = 1/2$, $r = 1/10$, $p = 2/5$.

(a) Draw the binary tree illustrating the possible paths followed by the stock price process.

(b) In your diagram, record the probabilities (when the “up” probability is $p$ and the “down” probability is $1 - p$) associated with the individual elements of the sample space $\Omega$.

(c) List the events making up the $\sigma$-field $F_1$ determined by $S_1$. (Be sure to include the empty set and the whole sample space.)

(d) Indicate on your binary tree the values (one for each path) of a European contingent claim whose payoff at $T = 3$ is $X = \max(S_0, S_1, S_2, S_3)$.

Solution. (a) The possible values for $S_3$ in the binary tree are (in order from left to right, left corresponding to $d$): 1, 6, 6, 36, 6, 36, 36, 216.

(b) The respective probabilities of the end nodes listed above are:

$$27/125, 18/125, 18/125, 12/125, 18/125, 12/125, 12/125, 8/125.$$  

(c) Let us list the elements of the sample space as follows:

$$\Omega = \{ddd, ddu, dud, duu, udd, udu, uud, uuu\}.$$  

With this labelling, the $\sigma$-field $F_1$ can be listed as follows:

$$F_1 = \{\emptyset, \Omega, \{ddd, ddu, dud, duu\}, \{udd, udu, uud, uuu\}\}.$$  

(d) Listing things in the same order as before, the possible values of $X$ are

$$8, 8, 12, 36, 24, 36, 72, 216.$$ 

3. (Exercise 2, Section 2.4)

Solution. In this exercise we have $T = 2$, $S_0 = $100, $u = 2$, $d = 1/2$, $r = 1/10$. The contingent claim under consideration is a European call option with strike $K = $80. Notice that the “risk-neutral” up probability is $p^* = 2/5$.

(a) The possible values of $X$ are 0, 20, and 320 dollars, with respective $P^*$ probabilities $9/25$, $12/25$, $4/25$. Therefore

$$V_0 = \frac{1}{(1+r)^2} \cdot E^*[X] = \left( \frac{10}{11} \right)^2 \cdot \left[ 0 \cdot \frac{9}{25} + 20 \cdot \frac{12}{25} + 320 \cdot \frac{4}{25} \right]$$

$$= \frac{100 \cdot 1520}{121 \cdot 25} = \frac{6080}{121}.$$ 

(b) Using the formulas (2.26) and (2.27) on page 15 of the text, we compute

$$\alpha_u^2 = 1, \quad \beta_u^2 = -\frac{8000}{121},$$

$$\alpha_d^2 = \frac{4}{15}, \quad \beta_d^2 = -\frac{2000}{363},$$

and

$$\alpha_1 = \frac{4}{5}, \quad \beta_1 = -\frac{10800}{363}.$$ 

As a reality check

$$\alpha_1 S_0 + \beta_1 B_0 = \frac{4}{5} \cdot 100 - \frac{10800}{363} = 80 - \frac{10800}{363} = \frac{29040 - 10800}{363} = \frac{18240}{363} = \frac{6080}{121},$$

consistent with part (a).

(c) Suppose that $C_0$ (the price of the option $X$) is equal to $V_0 - 2$ dollars. One arbitrage is to buy the option, sell the hedging portfolio, and put the $2 netted in these transactions into the bond. That is, we employ the trading strategy $\psi = (-\alpha_t, -\beta_t + 2, 1)$. This represents an initial investment of

$$V_0(\psi) = -\alpha_1 S_0 - \beta_1 + 2 + C_0 = -V_0 + 2 + C_0 = 0,$$

with time 2 value equal to

$$V_2(\psi) = -\alpha_2 S_2 - \beta_2 B_2 + 2B_2 + X = -X + 2B_2 + X = 2B_2 > 0,$$

a risk-free (strictly positive) profit.