1. Exercise 4, Section 2.4.

Solution. Under the risk-neutral probability \( P^* \), the discounted stock price process

\[
S_t^* = (1 + r)^{-t} S_t, \quad t = 0, 1, 2, \ldots, T,
\]

is a martingale:

\[
\mathbb{E}^*[S_{t+1}^* | \mathcal{F}_t] = S_t^*, \quad t = 0, 1, 2, \ldots, T - 1.
\]

Consequently, since \( V_t(\phi^*) = \alpha_t^* S_t + \beta_t^* B_t \),

\[
\mathbb{E}^*[V_{t+1}(\phi^*) | \mathcal{F}_t] = \mathbb{E}^*[\alpha_{t+1}^* S_{t+1} + \beta_{t+1}^* B_{t+1} | \mathcal{F}_t]
\]

\[
= \alpha_{t+1}^* \mathbb{E}^*[S_{t+1} | \mathcal{F}_t] + \beta_{t+1}^* B_{t+1}
\]

\[
= \alpha_{t+1}^*(1 + r) S_t^* + \beta_{t+1}^*(1 + r) B_t
\]

\[
= (1 + r) [\alpha_{t+1}^* S_t^* + \beta_{t+1}^* B_t]
\]

\[
= (1 + r) V_t(\phi^*).
\]

In this computation, the second equality follows because \( \alpha_{t+1}^* \) and \( \beta_{t+1}^* \) are \( \mathcal{F}_t \) measurable, the third equality follows from the martingale property of \( S_t^* \) while the fifth follows from the self-financing nature of the replicating strategy \( \phi^* \). In other words, the discounted value process

\[
V_t^*(\phi^*), \quad t = 0, 1, 2, \ldots, T,
\]

is a \( P^* \)-martingale. In particular, as with any martingale,

\[
V_t^*(\phi^*) = \mathbb{E}^*[V_T^*(\phi^*) | \mathcal{F}_t], \quad t = 0, 1, 2, \ldots, T,
\]

or, what amounts to the same thing,

\[
V_t(\phi^*) = (1 + r)^t \mathbb{E}^* \left[ \frac{V_T(\phi^*)}{(1 + r)^T} \bigg| \mathcal{F}_t \right] = \frac{1}{(1 + r)^{T-t}} \mathbb{E}^*[X | \mathcal{F}_t], \quad t = 0, 1, 2, \ldots, T,
\]

in which we used the fact that \( V_T(\phi^*) = X \). This is formula (2.65).
Suppose that we can trade in stock, bond, and contingent claim. A trading strategy is now a sequence of triples of random variables

$$\psi = \{(\alpha_t, \beta_t, \gamma_t) : t = 1, 2, \ldots, T\},$$

such that $\alpha_t, \beta_t, \gamma_t$ are all $\mathcal{F}_{t-1}$-measurable for $t = 1, 2, \ldots, T$, and

$$\alpha_t S_t + \beta_t B_t + \gamma_t C_t = \alpha_{t+1} S_t + \beta_{t+1} B_t + \gamma_{t+1} C_t, \quad t = 1, 2, \ldots, T - 1;$$

that is, the strategy is self-financing. (Here $C_t$ is the (possibly random) market price of the contingent claim at time $t$.) The value of such a trading strategy (at time $t \geq 1$) is $V_t(\psi) = \alpha_t S_t + \beta_t B_t + \gamma_t C_t$ (with $V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0$).

In this context, an arbitrage is a trading strategy $\psi$ such that (i) $V_0(\psi) = 0$, (ii) $V_T(\psi) \geq 0$, and (iii) $P^*[V_T(\psi) > 0] > 0$ (equivalently, $E^*[V_T(\psi)] > 0$). We shall now verify that $C_t = V_t(\phi^*)$, $t = 0, 1, 2, \ldots, T$, is the unique no-arbitrage price for the contingent claim in the extended market. (As before, $\phi^*$ is the replicating strategy for the claim $X$.)

First suppose that there is a time $u$ in $\{0, 1, 2, \ldots, T\}$ such that the event $G = \{C_u > V_u(\phi^*)\}$ satisfies $P^*[G] > 0$. Consider the following strategy: At time $u$, and if $G$ occurs, sell a unit of the option for $C_u$, use $V_u(\phi^*)$ of the proceeds to invest in the stock/bond market, and put the difference $C_u - V_u(\phi^*)$ into the bond. More precisely, consider the trading strategy $\psi$ defined by

$$\alpha_t = \beta_t = \gamma_t = 0, \quad t = 1, 2, \ldots, u - 1.$$

Moreover, if the event $G$ occurs then

$$\alpha_t = \alpha^*_t, \beta_t = \beta^*_t + (C_u - V_u(\phi^*))B_u^{-1}, \gamma_t = -1, \quad t = u, u + 1, \ldots, T,$$

while if $G^c$ occurs then

$$\alpha_t = \beta_t = \gamma_t = 0, \quad t = u, u + 1, \ldots, T,$$

Then $V_t(\psi) = 0$ for $t = 0, 1, 2, \ldots, u$, and

$$V_T(\psi) = \alpha^*_T S_T + \beta^*_T B_T + (C_u - V_u(\phi^*))(1 + r)^{T-u} - X$$

$$= X + (C_u - V_u(\phi^*))(1 + r)^{T-u} - X = (C_u - V_u(\phi^*))(1 + r)^{T-u} > 0,$$
yielding a risk-free profit. By the no-arbitrage principle we must therefore have

\[ P^*[C_u \leq V_u(\phi^*)], \text{ for all } u, 0, 1, 2, \ldots, T = 1. \]

Similarly, an arbitrage can be arranged if \( P^*[C_u < V_u(\phi^*)] > 0 \) for some \( u \). Consequently,

\[ P^*[C_u \geq V_u(\phi^*), \text{ for all } u, 0, 1, 2, \ldots, T = 1. \]

Combining these observations we see that

\[ P^*[C_u = V_u(\phi^*), \text{ for all } u, 0, 1, 2, \ldots, T = 1. \]

It remains to show that in this case, no arbitrage is possible.

Thus we suppose that \( C_t = V_t(\phi^*) \) for all \( t \in \{0, 1, 2, \ldots, T\} \). Let \( \psi \) be any (self-financing) trading strategy in the stock/bond/contingent claim market and suppose that \( V_0(\psi) = 0 \) but \( V_T(\psi) \geq 0 \). We compute:

\[
E^*[V_T(\psi)] = E^*[\alpha_T S_T + \beta_T B_T + \gamma_T C_T]
\]

\[
= (1 + r)E^*[\alpha_T S_{T-1} + \beta_T B_{T-1}] + E^*[\gamma_T E^*[C_T|F_{T-1}]]
\]

\[
= (1 + r)E^*[\alpha_T S_{T-1} + \beta_T B_{T-1}] + E^*[\gamma_T E^*[V_T(\phi^*)|F_{T-1}]]
\]

\[
= (1 + r)E^*[\alpha_T S_{T-1} + \beta_T B_{T-1}] + (1 + r)E^*[\gamma_T V_{T-1}(\phi^*)]
\]

\[
= (1 + r)E^*[\alpha_T S_{T-1} + \beta_T B_{T-1}] + (1 + r)E^*[\gamma_T C_{T-1}]
\]

\[
= (1 + r)E^*[V_{T-1}(\psi)]
\]

Proceeding recursively, we eventually find that

\[ E^*[V_T(\psi)] = (1 + r)^T E^*[V_0(\psi)] = 0. \]

This forces \( V_T(\psi) = 0 \) with \( P^* \) probability 1. Thus, no arbitrage is possible.

2. Exercise 5, Section 2.4.

Solution. Observe that if \( b \) is a real number then \( b^+ - (-b)^+ = b \). (Consider the cases \( b > 0 \) and \( b \leq 0 \) separately.) Thus

\[ C_T - P_T = (S_T - K)^+ - (K - S_T)^+ = S_T - K. \]

Combining this with the result of the preceding exercise,

\[
C_t - P_t = (1 + r)^{t-T} E^*[C_T - P_T|F_t]
\]

\[
= (1 + r)^{t-T} E^*[S_T - K|F_t]
\]

\[
= (1 + r)^{t-T} (E^*[S_T|F_t] - K)
\]

\[
= (1 + r)^{t-T} ((1 + r)^{T-t}S_t - K)
\]

\[
= S_t - (1 + r)^{t-T} K.
\]