1. For this problem use the model parameters of Exercise 2, section 2.4.

(a) Find the superhedging strategy for the American call option, and the time-zero cost of that strategy.

(b) Find the superhedging strategy for the American put option, and the time-zero cost of that strategy.

Solution. In this exercise we have $T = 2$, $S_0 = 100$, $u = 2$, $d = 1/2$, $r = 1/10$, and $K = 80$. The “risk-neutral” up probability is $p^* = 2/5$.

(a) The contingent claim tree has entries:

$$Y_0 = 20; Y_1^d = 0, Y_1^u = 120; Y_2^{dd} = 0, Y_2^{du} = 20, Y_2^{uu} = 320.$$

Using the $Y_2$ values to fill in the bottom row of the $U$ tree (the necessary wealth process) and then filling in the rest of the tree recursively we obtain

$$U_1^d = 80/11, \quad U_1^u = 1400/11,$$

and finally

$$U_0 = 6080/121 = 50.25$$

dollars. This is the no-arbitrage price of the American call option associated with the given data.

Using the formulas on page 21 of the text, we compute the superhedging strategy $\phi^* = (\alpha^*, \beta^*)$:

$$\alpha_{2, d}^* = \frac{20 - 0}{3/2} \cdot \frac{1}{50} = \frac{4}{15}, \quad \beta_{2, d}^* = \frac{2 \cdot 0 - .5 \cdot 20}{3/2} \cdot \left(\frac{10}{11}\right)^2 = -\frac{2000}{363},$$

$$\alpha_{2, u}^* = \frac{320 - 20}{3/2} \cdot \frac{1}{200} = 1, \quad \beta_{2, u}^* = \frac{2 \cdot 20 - .5 \cdot 320}{3/2} \cdot \left(\frac{10}{11}\right)^2 = -\frac{8000}{121},$$

and

$$\alpha_1^* = \frac{1400/11 - 80/11}{3/2} \cdot \frac{1}{100} = \frac{4}{5}, \quad \beta_1^* = \frac{2 \cdot (80/11) - .5 \cdot (1400/11) \cdot 10}{3/2} = -\frac{3600}{121}.$$

Notice that $\tilde{\delta}_1 = \tilde{\delta}_2 = 0$ in the present example.
(b) The contingent claim tree has entries:

\[ Y_0 = 0; Y_1^d = 30, Y_1^u = 0; Y_2^{dd} = 55, Y_2^{du} = Y_2^{ud} = Y_2^{uu} = 0. \]

Using the \( Y_2 \) values to fill in the bottom row of the \( U \) tree (the necessary wealth process) and then filling in the rest of the tree recursively we obtain

\[ U_1^d = 30, \quad U_1^u = 0, \]

and finally

\[ U_0 = 180/11 = 16.36 \]

dollars. This is the no-arbitrage price of the American put option associated with the given data.

Using the formulas on page 21 of the text, we compute we compute the superhedging strategy \( \phi^* = (\alpha^*, \beta^*) \):

\[
\alpha_2^{*,d} = \frac{-55}{3/2} \cdot \frac{1}{50} = -\frac{11}{15}; \quad \beta_2^{*,d} = \beta_2^{d} = \frac{2 \cdot 55 - 5 \cdot 0}{10} \cdot \left( \frac{10}{11} \right)^2 = \frac{2000}{33},
\]

\[ \alpha_2^{*,u} = 0, \quad \beta_2^{*,u} = \beta_2^{u} = 0, \]

and

\[
\alpha_1^* = \frac{-30}{3/2} \cdot \frac{1}{100} = -\frac{1}{5}; \quad \beta_1^* = \beta_1 = \frac{2 \cdot 30}{3/2} \cdot \frac{10}{11} = \frac{400}{11}.
\]

Notice that \( \tilde{\delta}_1 = \tilde{\delta}_2 = 0 \) in the present example.

2. Exercise 6, Section 2.4 (pages 29–30 of the text), but do not do part (d).

Solution. In this exercise we also have \( T = 2, S_0 = \$100, u = 2, d = 1/2, r = 1/10 \). The “risk-neutral” up probability is again \( p^* = 2/5 \). The contingent claim is an American put option with strike \( K = \$120 \).

(a) The contingent claim tree has entries:

\[ Y_0 = 20; Y_1^d = 70, Y_1^u = 0; Y_2^{dd} = 95, Y_2^{du} = Y_2^{ud} = 20, Y_2^{uu} = 0. \]

Using the \( Y_2 \) values to fill in the bottom row of the \( U \) tree (the necessary wealth process) and then filling in the rest of the tree recursively we obtain

\[ U_1^d = 70, \quad U_1^u = 120/11, \]
and finally

\[ U_0 = \frac{5100}{121} = 42.15 \]

dollars. This is the no-arbitrage price of the American put option associated with the given data.

(b) Using the formulas on page 21 of the text, we compute the super-hedging strategy \( \phi^* = (\alpha^*, \beta^*) \):

\[ \alpha^*_{d,2} = \frac{20 - 95}{3/2} \cdot \frac{1}{50} = -1, \quad \tilde{\beta}^d_2 = \frac{2 \cdot 95 - .5 \cdot 20}{3/2} \cdot \left( \frac{10}{11} \right)^2 = \frac{12000}{121} \]

and

\[ \tilde{\delta}^d_2 = 70 - \frac{650}{11} = \frac{120}{11}. \]

Thus

\[ \beta^*_{d,2} = \tilde{\beta}^d_2 + \tilde{\delta}^d_2 / B_1 = \frac{12000}{121} + \left( \frac{120}{11} \right) \left( \frac{10}{11} \right) = \frac{13200}{121} = \frac{1200}{11}. \]

Also,

\[ \alpha^*_{u,2} = \frac{0 - 20}{3/2} \cdot \frac{1}{200} = -\frac{1}{15}, \quad \beta^*_{u,2} = \tilde{\beta}^u_2 = \frac{2 \cdot 20 - .5 \cdot 0}{3/2} \cdot \left( \frac{10}{11} \right)^2 = \frac{8000}{363}, \]

and

\[ \alpha^*_1 = \frac{120/11 - 770/11}{3/2} \cdot \frac{1}{100} = -\frac{13}{33}, \quad \beta^*_1 = \tilde{\beta}^1_1 = \frac{2 \cdot 70 - .5 \cdot (120/11)}{3/2} \cdot \frac{10}{11} = \frac{29600}{363}. \]

Notice that \( \tilde{\delta}_1 = 0 = \tilde{\delta}^u_2 \) but \( \tilde{\delta}^d_2 > 0 \) in the present example.

(c) Sell the superhedging strategy \( \phi^* \) for 42.15 dollars, buy the put option for 41.15 dollars, and invest the remaining $1 in the bond. Use the stopping time \( \tau^* \) that is equal to 2 if \( S_1 = 200 \) and equal to 1 if \( S_1 = 50 \). As in the discussion in the second paragraph on page 27 of the text, these actions provide for a risk-free profit of \( (11/10)^2 > (11/10)^2 = 1.21 \) dollars.