Math 194, Winter 2020

Homework 7 — Due Tuesday, February 25, 2020

1. Consider the multi-step binary model with $T = 2$, $S_0 = \$100$, $u = 2$, $d = 1/2$, and $r = 1/5$. In this problem we shall use the real-world probability $\mathbf{P}$ for which "up" and "down" branches each have probability $p = 1/2$. Let adapted sequences of random variables $Y = \{Y_0, Y_1, Y_2\}$ and $U = \{U_0, U_1, U_2\}$ be defined by

$$Y_0 = 30, Y_1^d = 24, Y_1^u = 34, Y_2^{dd} = Y_2^{uu} = 50, Y_2^{du} = Y_2^{ud} = 10,$$

and

$$U_0 = 32, U_1^d = 30, U_1^u = 34, U_2^{dd} = U_2^{uu} = 50, U_2^{du} = U_2^{ud} = 10.$$

(a) Verify that $\{U_0, U_1, U_2\}$ is a supermartingale.

(b) Evidently $U_t \ge Y_t$ for $t = 0, 1, 2$. Show that if $W = \{W_0, W_1, W_2\}$ is another supermartingale with $W_t \ge Y_t$ for each $t = 0, 1, 2$, then $W_t \ge U_t$ for $t = 0, 1, 2$.

(c) The recipe $\tau = \min\{t : U_t = Y_t\}$ defines a stopping time. Compute $\tau(dd)$, $\tau(du)$, $\tau(ud)$, and $\tau(uu)$.

(d) Compute $\mathbf{E}[Y_\tau]$, where τ is as in part (c).

2. For this problem use the model data from Exercise 6, Section 2.4 (page 29 of the text). An American *asset-or-nothing* (call) option is the American contingent claim (ACC) based on the payoffs

$$Y_t = \begin{cases} F, & \text{if } S_t \ge K, \\ 0, & \text{if } S_t < K, \end{cases}$$

and can be exercised at any time t in $\{0, 1, 2\}$. Let us take $K = \$90$ and $F = \$120$. Fill in the tree diagram for the necessary wealth process $U = \{U_0, U_1, U_2\}$, thereby computing the no-arbitrage price U_0 for this ACC.

3. Exercise 2, Section 3.7 (pages 52–53 of the text).