1. In this exercise we use the set-up of Exercise 2, Section 3.7 (page 52–53 of the text). Consult the solution of Exercise 3 on Homework 7 for a description of an Equivalent Martingale Measure $\mathbf{P}^*$.

(a) Use $\mathbf{P}^*$ to compute the time $t$ price $C_t$ of the European call option with payoff $(S_2^1 - 6)^+$ for $t = 0,1$.

(b) Now use $\mathbf{P}^*$ to compute the time $t$ price $P_t$ of the European put option with payoff $(6 - S_2^1)^+$ for $t = 0,1$.

(c) Confirm the call-put parity relationship $C_t - P_t = S_t - 6$ for $t = 0,1$.

Solution. (a) The no-arbitrage price process $C_t$ can be computed by filling in the tree obtained from backward recursion using $C_2 = (S_2^1 - 6)^+$ in the bottom row of the tree. We obtain

$$C_{2dd} = C_{2du} = C_{2ud} = 0, \quad C_{2uu} = 3,$$

$$C_1^d = 0, \quad C_1^u = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 3 = 2,$$

and

$$C_0 = \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

(b) The no-arbitrage price process $P_t$ can be computed by filling in the tree obtained from backward recursion using $P_2 = (6 - S_2^1)^+$ in the bottom row of the tree. We obtain

$$P_{2dd} = 4, P_{2du} = 1, P_{2ud} = 0 = P_{2uu} = 0,$$

$$P_1^d = \frac{1}{3} \cdot 4 + \frac{2}{3} \cdot 1 = 2, \quad P_1^u = 0,$$

and

$$P_0 = \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 0 = \frac{3}{2}.$$

(c) Subtracting $P_t$ from $C_t$ we obtain

$$C_0 - P_0 = -1,$$

$$C_1^d - P_1^d = -2, \quad C_1^u - P_1^u = 2,$$

and

$$C_{2dd} - P_{2dd} = -4, \quad C_{2du} - P_{2du} = -1, \quad C_{2ud} - P_{2ud} = 0, \quad C_{2uu} - P_{2uu} = 3.$$

On the other hand,

$$S_0^1 = 5,$$


\[ S_1^{1,d} = 4, \quad S_1^{1,u} = 8, \]

and

\[ S_2^{1,dd} = 2, \quad S_2^{1,du} = 5, \quad S_2^{1,ud} = 6, \quad S_2^{1,uu} = 9, \]

so that

\[ S_0^1 - 6 = -1, \]
\[ S_1^{1,d} - 6 = -2, \quad S_1^{1,u} - 6 = 2, \]

and

\[ S_2^{1,dd} - 6 = -4, \quad S_2^{1,du} - 6 = -1, \quad S_2^{1,ud} - 6 = 0, \quad S_2^{1,uu} - 6 = 3, \]

This shows that \( C_t(\omega) - P_t(\omega) = S_t^1(\omega) - K \) for each \( t = 0, 1, 2 \) and each sample point \( \omega \in \Omega \), confirming call-put parity.

2. In this exercise we also use the set-up of Exercise 2, Section 3.7 (page 52–53 of the text). In Homework 7 you saw that this model admits an Equivalent Martingale Measure. This market is therefore viable, by the First Fundamental Theorem of Asset Pricing. Is this market complete?

Solution. Yes: The computations performed in finding \( P^* \) in Exercise 3 of Homework 7 showed that \( P^* \) was uniquely determined by the data (i.e., by the stock-price tree). In view of The Second Fundamental Theorem of Asset Pricing (Theorem 3.3.2, page 41), the market is therefore complete.

3. Exercise 3, Section 3.7 (page 53 of the text).

Solution. (b) We proceed as in the solution of Exercise 3 of Homework 7. As there we can use the familiar formula for the risk-neutral “up” probability to compute:

\[ p^0 = P^*[S_1^1 = 8] = \frac{1 + .1 - (4/5)}{8/5 - 4/5} = \frac{3}{8}, \]

and similarly

\[ p^{1,u} = P[S_2^1 = 9|S_1^1 = 8] = \frac{9}{10}, \]

and these two “branch” probabilities are uniquely determined by the data. The probabilities for the remaining three-way branch exist, but are not uniquely determined. Indeed, writing

\[ a = P^*[S_2^1 = 2|S_1^1 = 4], \quad b = P^*[S_2^1 = 5|S_1^1 = 4], \quad c = P^*[S_2^1 = 6|S_1^1 = 4], \]

we must have

\[ a + b + c = 1, \]
\[ (2a + 5b + 6c) \cdot \frac{10}{11} = 4 \]
(the martingale property of the discounted stock-price process), and

\[ a > 0, \quad b > 0, \quad c > 0. \]

The general solution of this constrained linear system (when parametrized by \( c \)) is

\[
\left( \frac{1}{5} + \frac{c}{3}, \frac{4}{5} - \frac{4c}{3}, c \right), \quad 0 < c < \frac{3}{5}.
\]

So equivalent martingale measures exist, but are not unique.

If \( r = 1 \), then there is an arbitrage opportunity, hence no EMM. Indeed, suppose that at time 0 we (short) sell one share of the risky asset and invest the proceeds in the non-risky investment. For this portfolio \( \phi_1^0 = \phi_2^0 = 5, \phi_1^1 = \phi_2^1 = -1 \), \( V_0(\phi) = 0 \), and

\[ V_2(\phi) = 20 - S_2^1, \]

and this final quantity is strictly positive: it is greater than or equal to 11 since \( S_2^1 \leq 9 \) in all cases. Thus \( \phi \) represents an arbitrage opportunity.