1. Give a definition of each of the following terms (in the context of the multi-period binomial model):

(a) arbitrage opportunity
(b) European call option with strike price \( K \) dollars
(c) trading strategy

**Solution.** Consult the text.

2. Consider the multi-period binomial model with \( T = 2, u = 2, d = 1/2, r = 0, \) and \( S_0 = 40. \) The hedging strategy \( \phi = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} \) for the call option \( X = (S_2 - 20)^+ \) is given by

\[
\alpha_1 = \frac{8}{9}, \quad \beta_1 = -\frac{100}{9}, \\
\alpha_2^d = \frac{2}{3}, \quad \beta_2^d = -\frac{20}{3}, \quad \alpha_2^u = 1, \quad \beta_2^u = -20.
\]

(a) Verify that \( \phi \) is self-financing: That is, show that \( \alpha_1 S_1 + \beta_1 B_1 = \alpha_2 S_1 + \beta_2 B_1. \) (Be sure to consider both possibilities for \( S_1. \))

(b) Compute \( V_0(\phi). \)

**Solution.** (a) Let us verify that \( \alpha_1 S_1 + \beta_1 B_1 = \alpha_2 S_1 + \beta_2 B_1 \) by considering the up and down cases separately. In the “up” case we have

\[
\alpha_1 S_1 + \beta_1 B_1 = \alpha_1 \cdot 80 + \beta_1 \\
= \frac{8}{9} \cdot 80 - \frac{100}{9} \\
= \frac{540}{9} = 60,
\]

while

\[
\alpha_2 S_1 + \beta_2 B_1 = 1 \cdot 80 - 20 \\
= 60,
\]

which yields the desired equality. In the “down” case we have

\[
\alpha_1 S_1 + \beta_1 B_1 = \frac{8}{9} \cdot 20 - \frac{100}{9} = \frac{160 - 100}{9} = \frac{60}{9} = \frac{20}{3},
\]

while

\[
\alpha_2 S_1 + \beta_2 B_1 = \frac{2}{3} \cdot 20 - \frac{20}{3} = \frac{20}{3}.
\]

Thus we have the desired equality in the down case as well.

(b) \( V_0(\phi) = \alpha_1 S_0 + \beta_1 = (8/9)40 - (100/9) = 220/9. \) As a check, because \( r = 0, \) we also have

\[
V_0(\phi) = \mathbf{E}^*[X] = \frac{4}{9} \cdot 0 + \frac{2}{9} \cdot 20 + \frac{2}{9} \cdot 20 + \frac{1}{9} \cdot 140 \\
= \frac{220}{9}.
\]
3. Consider the single-period binomial model with $S_0 = $200, $S_1 = $400 or $100, $r = 0.2$.

(a) Find the no-arbitrage initial price of a European put option (on one share of stock) with strike price $K = $200.

(b) Find a hedging strategy that replicates the option described in (a).

(c) Suppose the market price of the option in (a) is $1 less than the no-arbitrage price. Describe a strategy (for trading in stock, bond, and the option) that yields an arbitrage opportunity.

Solution. (a) The put option $X = (200 - S_1)^+$ takes on the two possible values 100 and 0 with $P^*$ probabilities $1 - p^* = 8/15$ and $p^* = 7/15$, respectively. Consequently, the no-arbitrage price of $X$ is

$$V_0 = \frac{1}{1 + r}E^*[X] = \frac{5}{6} \left[ \frac{8}{15} \cdot 100 + \frac{7}{15} \cdot 0 \right] = \frac{400}{9} \text{ dollars.}$$

(b) The hedging strategy is given by

$$\alpha_1 = \frac{0 - 100}{2 - .5} \cdot \frac{1}{200} = \frac{1}{3}, \quad \beta_1 = \frac{2 \cdot 100 - .5 \cdot 0}{2 - .5} \cdot \frac{6}{6} = \frac{1000}{9}.$$

Check: $V_0(\phi) = \alpha_1 S_0 + \beta_1 B_0 = (-1/3)200 + 1000/9 = 400/9$, as expected.

(c) If $C_0$ (the price of the put option) is equal to $V_0 - 1$, then we buy the option, sell the hedging portfolio, and put the remaining dollar in the bond. That is, we use the trading strategy $\psi = (-\alpha_1, -\beta_1 + 1, 1)$, where $\alpha_1$ and $\beta_1$ are as in part (b). Then

$$V_0(\psi) = -V_0(\phi) + 1 + C_0 = 0,$$

while

$$V_1(\psi) = -V_1(\phi) + (6/5) + X = -X + (6/5) + X = 6/5 > 0,$$

so that $\psi$ presents an arbitrage.

4. For the CCR model with parameters $T = 2$, $u = 3$, $d = 1/3$, and $r = 0$, the “risk-neutral” up probability is $p^* = 1/4$. Three random variables $M_0$, $M_1$, and $M_2$ are defined on the corresponding sample space by $M_0 = 7/4$ (constant),

$$M_1(dd) = M_1(du) = 5/4, \quad M_1(ud) = M_1(uu) = 13/4,$$

and

$$M_2(dd) = 1, \quad M_2(du) = 2, \quad M_2(ud) = 3, \quad M_2(uu) = 4.$$

(It may be helpful to draw the binary tree.)

(a) Verify that $\{M_0, M_1, M_2\}$ is a martingale with respect to the probability $P^*$.

(b) Compute $E^*[M_2]$.

Solution. (a) We need to show that $E^*[M_1|\mathcal{F}_0] = M_0$ and $E^*[M_2|\mathcal{F}_1] = M_1$. 

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Because \( \mathcal{F}_0 = \{\emptyset, \Omega\} \), the first assertion amounts to showing that \( \mathbb{E}^*[M_1] = M_0 \). Computing:

\[
\mathbb{E}^*[M_1] = \frac{3}{4} \cdot M_1^d + \frac{1}{4} \cdot M_1^u = \frac{3}{4} \cdot 5 + \frac{1}{4} \cdot 13 = \frac{15 + 13}{16} = \frac{28}{16} = \frac{7}{4} = M_0.
\]

For \( \mathbb{E}^*[M_2|\mathcal{F}_1] \) there are two cases. In the up case, the conditional expectation is the weighted average of \( M_2(ud) \) and \( M_2(uu) \) with respective weights \( 3/4 \) and \( 1/4 \). This average is

\[
\frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 4 = \frac{13}{4},
\]

which coincides with \( M_1^u \), the given value \( M_1(ud) = M_1(uu) \) of \( M_1 \), in the up case. In the down case, the conditional expectation is the weighted average of \( M_2(dd) \) and \( M_2(du) \), again with respective weights \( 3/4 \) and \( 1/4 \). This average is

\[
\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 2 = \frac{5}{4},
\]

which coincides with \( M_1^d \), the given value \( M_1(dd) = M_1(du) \) of \( M_1 \), in the down case. This shows that \( \mathbb{E}^*[M_2|\mathcal{F}_1] = M_1 \) in all cases.

(b) Direct calculation yields

\[
\mathbb{E}^*[M_2] = \frac{9}{16} \cdot 1 + \frac{3}{16} \cdot 2 + \frac{3}{16} \cdot 3 + \frac{1}{16} \cdot 4 = \frac{9 + 6 + 9 + 4}{16} = \frac{28}{16} = \frac{7}{4}.
\]

Of course, we could have saved ourselves some work by recalling that (because \( \{M_0, M_1, M_2\} \) is a martingale) \( \mathbb{E}^*[M_2] = M_0 \), which we know to be \( 7/4 \).