2. Consider the multi-period binomial model with \( T = 2, u = 2, d = 1/2, \) \( r = 1/5, \) and \( S_0 = 100. \) Your goal in this problem is to price an American put option with strike price \( K = 120 \) dollars.

(a) Make the tree diagram listing the possible values for the contingent claim \( Y_t = (120 - S_t)^+, \) \( t = 0, 1, 2. \)

(b) Compute the risk-neutral “up” probability \( p^* \), and then use backward recursion to fill in the tree recording the values of the necessary wealth process \( U = \{U_0, U_1, U_2\}. \)

(c) Find the time-zero price of this American put option with strike price \( K = $120 \)?

Solution. (a) We have

\[
\begin{align*}
Y_0 &= 20 \\
Y_{1d} &= 70, \quad Y_{1u} = 0, \\
Y_{2dd} &= 95, \quad Y_{2du} = Y_{2ud} = 20, \quad Y_{2uu} = 0.
\end{align*}
\]

(b) First, \( p^* = (1 + r - d)/(u - d) = (1.2 - .5)/1.5 = 7/15. \) Using this value for \( p^* \) and the discount factor \( 1/(1+r) = 5/6 \) and starting with

\[
U_{2dd} = 95, \quad U_{2du} = U_{2ud} = 20, \quad U_{2uu} = 0,
\]

we find that

\[
U_{1d} = \max \left( 70, \left( \frac{8}{15} \cdot 95 + \frac{7}{15} \cdot 20 \right) \cdot (5/6) \right) = \max(70, 50) = 70, \quad U_{1u} = \frac{8}{15} \cdot 20 \cdot (5/6) = \frac{80}{9},
\]

and

\[
U_0 = \max \left( 20, \left( \frac{8}{15} \cdot 70 + \frac{7}{15} \cdot \frac{80}{9} \right) \cdot (5/6) \right) = \frac{2800}{81} = 34.57.
\]

(c) In view of part (b), the time-zero price of this put option is $34.57.

3. In this problem \( T = 1, \) the sample space has just two points: \( \Omega = \{\omega_1, \omega_2\}, \) with \( P[\{\omega_1\}] = P[\{\omega_2\}] = 1/2. \) The value of the riskless security \( S_t^0 \) is equal to 1 for \( t = 0, 1. \) There are two risky securities given by \( S_t^1 = S_t^2 = 2 \) and

\[
\begin{align*}
S_t^1(\omega_1) &= 5, \quad S_t^1(\omega_2) = 1, \\
S_t^2(\omega_1) &= 1, \quad S_t^2(\omega_2) = 3,
\end{align*}
\]

(As usual, \( S_t^i \) is the price of security \( i \) at time \( t. \)) We have \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) and \( \mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2\}\}. \) An equivalent probability measure \( P^* \) on \( \Omega \) is determined by the single number \( p^* = P^*[\{\omega_1\}], \) with \( 0 < p^* < 1, \) for then \( P[\{\omega_2\}] = 1 - p^*. \)

(a) Show that there is no Equivalent Martingale Measure in the present situation. That is, there is no \( P^* \) such that both \( E^*[S_t^1] = S_0^1 \) and \( E^*[S_t^2] = S_0^2. \)
(b) Show that the trading strategy \( \phi = (\phi_0^1, \phi_1^1, \phi_1^2) \) given by \( \phi_0^1 = -4, \ \phi_1^1 = 1, \ \phi_1^2 = 1 \) represents an arbitrage opportunity.

Solution. (a) If there were an EMM \( P^* \), with “up” probability \( p^* \), then (because the bond price is constantly 1) we would have

\[
E^*[S^1_1|\mathcal{F}_0] = S^1_0, \quad E^*[S^2_1|\mathcal{F}_0] = S^2_0.
\]

Writing these equations out explicitly (remember that \( \mathcal{F}_0 \) is trivial, so \( E^*[X|\mathcal{F}_0] = E^*[X] \) for any random variable \( X \)) we arrive at the two conditions

\[
5 \cdot p^* + 1 \cdot (1 - p^*) = 2, \quad 1 \cdot p^* + 3 \cdot (1 - p^*) = 2.
\]

These two equations form an inconsistent system: The solution of the first equation is \( p^* = 1/4 \) while the solution of the second is \( p^* = 1/2 \). Thus no choice of \( p^* \) makes both (discounted) stock-price processes into martingales, so there is no EMM.

(b) We compute:

\[
V_0(\phi) = -4 + 1 \cdot 2 + 1 \cdot 2 = 0,
\]

\[
V_1(\phi)(\omega_1) = -4 + 1 \cdot 5 + 1 \cdot 1 = 2,
\]

and

\[
V_1(\phi)(\omega_2) = -4 + 1 \cdot 1 + 1 \cdot 3 = 0.
\]

Thus, \( V_0(\phi) = 0, V_1(\phi) \geq 0 \) and \( P[V_1(\phi) > 0] > 0 \). This means that \( \phi \) is an arbitrage opportunity.

[This is a confirmation of the First Fundamental Theorem of Asset Pricing: The presence of an arbitrage opportunity in the market precludes the existence of an Equivalent Martingale Measure.]

4. The tree for the binomial model with parameters \( T = 2, u = 2, d = 1, r = 1/4 \), is drawn below. We take \( \Omega = \{dd, du, ud, uu\} \), \( \mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_1 = \{\emptyset, \Omega, \{dd, du\}, \{ud, uu\}\} \), and \( \mathcal{F}_2 \) the class of all subsets of \( \Omega \). The risk neutral measure \( P^* \) is determined by the “up” probability \( p^* = 1/4 \). The nodes of the tree, as labelled, determine three random variables \( Z_0, Z_1, \) and \( Z_2 \). (For example, \( Z_2(dd) = 1, Z_2(ud) = 2, \) and \( Z_1(uu) = Z_1(ud) = 3. \) The sequence \( \{Z_0, Z_1, Z_2\} \) is a \( P^* \) supermartingale—you may take this assertion for granted!

(a) Compute \( E^*[Z_k] \) for \( k = 0, 1, 2, \) and verify numerically that \( E^*[Z_0] \geq E^*[Z_1] \geq E^*[Z_2] \).

(b) Consider the random variable \( \tau \) defined by \( \tau(dd) = \tau(du) = 1, \ \tau(ud) = \tau(uu) = 2 \). Verify that \( \tau \) is a stopping time.

(c) Compute \( E^*[Z_\tau] \).

Solution. (a) Since \( Z_0 = 2, \) we have \( E^*[Z_0] = 2. \) Next, \( Z_1 \) has possible values 1 and 3 with respective \( P^* \) probabilities 3/4 and 1/4. Therefore

\[
E^*[Z_1] = (3/4)1 + (1/4)3 = 1.5.
\]

Finally, \( Z_2 \) has possible values 1, 0, 2, 5, with respective \( P^* \) probabilities 9/16, 3/16, 3/16, 1/16. Therefore

\[
E^*[Z_2] = \frac{9 \cdot 1 + 3 \cdot 0 + 3 \cdot 2 + 1 \cdot 5}{16} = \frac{20}{16} = 1.25.
\]
These values confirm that $E^*[Z_0] \geq E^*[Z_1] \geq E^*[Z_2]$, as expected.

(b) Recall that a random variable $\tau$, with set of possible values $\{0, 1, 2\}$, is a stopping time provided $\{\omega \in \Omega : \tau(\omega) = t\} \in \mathcal{F}_t$ for $t = 0, 1, 2$. Let us check that this is the case for the given $\tau$.

Clearly $\{\tau = 0\} = \emptyset \in \mathcal{F}_0$,

$\{\tau = 1\} = \{dd, du\} \in \mathcal{F}_1$

(consult the listing of $\mathcal{F}_1$ in the statement of the problem), and

$\{\tau = 2\} = \{ud, uu\} \in \mathcal{F}_2$

(because $\mathcal{F}_2$ contains all subsets of $\Omega$). We have confirmed that $\tau$ is indeed a stopping time.

(c) Notice that $Z_\tau = 1$ on the event $\{\tau = 1\}$, which event has $P^*$ probability $3/4$. Similarly, $Z_\tau(ud) = 2$ (and $P^*\{ud\} = 3/16$) and $Z_\tau(uu) = 5$ (and $P^*\{uu\} = 1/16$). It follows that

$$E^*[Z_\tau] = \frac{3}{4} \cdot 1 + \frac{3}{16} \cdot 2 + \frac{1}{16} \cdot 5 = \frac{23}{16} = 1.4375.$$

(We have computed the numerical values

$$E[Z_0] = 2, \quad E[Z_\tau] = 1.4375, \quad E[Z_2] = 1.25,$$

so that

$$E[Z_0] > E[Z_\tau] > E[Z_2],$$

in consonance with Doob’s Stopping Theorem.)