Math 180B, Winter 2021

Homework 2, Due January 18

1. Use the formula $\mathbf{P}(A) = \mathbf{P}(A|B)\mathbf{P}(B) + \mathbf{P}(A|B^c)\mathbf{P}(B^c)$ to prove that if $\mathbf{P}(A|B) = \mathbf{P}(A|B^c)$ then A and B are independent. Then prove the converse (that if A and B are independent then $\mathbf{P}(A|B) = \mathbf{P}(A|B^c)$). [Assume that $\mathbf{P}(B) > 0$ and $\mathbf{P}(B^c) > 0$.]

2. Let X_1 and X_2 be the numbers showing when two fair dice are thrown. Define new random variables $X = X_1 - X_2$ and $Y = X_1 + X_2$. Show that X and Y are uncorrelated but not independent. [Hint: To show lack of independence, it is enough to show that $\mathbf{P}[X = j, Y = k] \neq \mathbf{P}[X = j] \cdot \mathbf{P}[Y = k]$ for *one* pair (j, k); try the pair $(0, 2)$.]

3. You have N boxes (labeled $1, 2, \ldots, N$), and you have k balls. You drop the balls, independently of each other, into the boxes. For each ball the probability that it will land in a particular box is $1/N$. Let X_1 be the number of balls in box 1 and X_N the number of balls in box N. Calculate $\text{Corr}(X_1, X_N)$.

4. Suppose X and Y are standard normal random variables. Find an expression for $\mathbf{P}(X + 2Y \leq 3)$ in terms of the standard normal distribution function Φ in two cases:

(a) X and Y are independent;

(b) X and Y have the bivariate normal distribution with correlation $\rho = 1/2$.

5. Let X_1 and X_2 be two independent standard normal random variables. Define two new random variables as follows: $Y_1 = X_1 + X_2$ and $Y_2 = X_1 + \beta X_2$. You are not given the constant β but it is known that $\text{Cov}(Y_1, Y_2) = 0$. Find

(a) the density of Y_2;

(b) $\text{Cov}(X_2, Y_2)$,

6. Suppose that (W, Z) have a bivariate normal distribution, that $W \sim \mathcal{N}(0, 1)$, and that the conditional distribution of Z, given that $W = w$, is $\mathcal{N}(aw + b, \tau^2)$.

(a) What is the marginal distribution of Z?

(b) What is the conditional distribution of W, given that $Z = z$?

In addition:

Pages 64–65:. Exercises 2.3.1, 2.3.5; Problems 2.3.2, 2.3.4(a)

Pages 70-71:. Exercise 2.4.3