Math 285, Winter 2023

Homework 5, Due February 16

1. Exercise 4.9 (pp. 99–100).

2. Here is a summary of the Viterbi algorithm:

(1) (Initialize): $\delta_0(j) = p(j)h_j(b_0)$ for $j \in S$.

(2) (Recursion): $\delta_{k+1}(j) = [\max_{i\in S} \delta_k(i)P(i,j)]\, h_j(b_{k+1})$, for $j \in S$ and
$\psi_{k+1}(j) = \text{argmax}_{i\in S}\delta_k(i)P(i,j)$ for $k = 0, 1, 2, \dots n-1$.

(3) (Termination): $P^*(b)[= P^*(b_0, \dots b_n)] = \max_{j\in S} \delta_n(j)$ and $j_n^* = \text{argmax}_{j\in S}\delta_n(j)$.

(4) (Back substitution): $j_k^* = \psi_{k+1}(j_{k+1}^*)$ for $k = n-1, n-2, \dots, 2, 1, 0$.

Show, by backward induction on k, that $p(j_k^*, j_{k+1}^*) > 0$ for $k = 0, 1, 2, \dots, n-1$.

3. This problems implements the Viterbi algorithm for the "crooked casino" HMM. The underlying Markov chain has state space $S = \{F, L\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} .95 & .05 \\ .1 & .9 \end{bmatrix}.$$

When the Markov chain is in state F, a fair die is thrown—all six faces are equally likely; when the Markov chain is in state L a loaded die is thrown—face 6 has probability .5 and all other faces have probability .1. In short, the "emission probabilities" are given by the matrix

$$\mathbf{H} = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/2 \end{bmatrix}.$$

We assume that at the start the fair die is being used; that is, $p(F) = 1$ and $p(L) = 0$.

Here are 100 random rolls in the crooked casino. Use the Viterbi algorithm to predict the state sequence; that is, which of the two dice was used on each roll.

3151162464 4664424531 1321631164 1521336251 4454363165
6626566666 6511664531 3265124563 6664631636 6631623264

On the second page are the dice that are tossed on each roll. Don't look until you have done the exercise!

Here are the dice that are rolled, each listed just below the roll itself.

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3151162464 4664424531 1321631164 1521336251 4454363165
FFFFFFFFFF FFFFFFFFFF FFFFFFFFFF FFFFFFFFFF FFFFFLLLLL
6626566666 6511664531 3265124563 6664631636 6631623264
LLLLLLLLLL LLLLLLFFFF FFFFFFFFLL LLLLLLLLLL LLLLFFFLLL
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