A NOTE ON “A NOTE ON THE EQUILIBRIUM
POTENTIAL OF CERTAIN DIRICHLET SPACES”

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We give a probabilistic proof of the main result of [1]. Let \( X = (X_t, \mathbb{P}^x) \) be a symmetric Markov process. We employ the usual notation.

Let \( B \) be a Borel subset of \( E \) with finite capacity, and let \( \pi \) denote the equilibrium measure for \( B \). Thus, the hitting probability \( \varphi(x) := \mathbb{P}^x(T_B < \infty) (x \in E) \) is an excessive \( m \)-version of the equilibrium potential \( U(\pi) \). In fact,

\[
P^x(f(X_{L_B}^-); L_B > 0) = U(f \cdot \pi)(x)
\]

for q.e. \( x \in E \), for each positive Borel function \( f \). Here \( L_B \) is the last exit time from \( B \), so that \( \{L_B > 0\} = \{T_B < \infty\} \).

**Lemma.** Let \( G \in \mathcal{E}^c \) be finely open, and define \( g(x) := \mathbb{E}^x(\exp(-T_{G^c})) \). Then \( g \) is finely continuous, and

\[
\{t \geq 0 : X_t \in G\} = \{t \geq 0 : g(X_t) < 1\}
\]

\( \mathbb{P}^x \)-a.s. for q.e. \( x \in E \).

**Proof.** The function \( g \) is 1-excessive, hence finely continuous. Evidently \( G \subset \{g < 1\} \). Moreover, \( \{g < 1\} \setminus G \) is the set of points of \( G^c \) that are irregular for \( G^c \). This set is semipolar, hence \( m \)-polar. 0

Suppose that \( f \) in (1) vanishes outside the fine interior \( B^{s^f} \) of \( B \). Let \( g \) be the function in the Lemma corresponding to \( G = B^{s^f} \). If \( X_{L_B}^- \in B^{s^f} \) and \( X_{L_B}^- = X_{L_B} \) then \( g(X_{L_B}) < 1 \), so \( g(X) < 1 \) on \( [L_B, L_B + \epsilon] \) for some (random) \( \epsilon > 0 \), because \( t \mapsto g(X_t) \) is almost surely right continuous. But this implies that \( X \) is in \( B^{s^f} \subset B \) after the last exit time from \( B \), which is absurd. It follows that if \( X_{L_B}^- \in B^{s^f} \) then \( X_{L_B} \notin B^{s^f} \); in particular, \( L_B \) is a jump time of \( X \) on \( \{X_{L_B}^- \in B^{s^f}\} \). Thus, for q.e. \( x \in E \),

\[
P^x(f(X_{L_B}^-); L_B > 0)
= \mathbb{P}^x \sum_{t \in J \cup \zeta} f(X_{t^-}) 1_{[X_{t^-} \in B^{s^f}]} 1_{[T_{B} \in \theta_t = \infty]} + \mathbb{P}^x(f(X_{\zeta^-}) 1_{[X_{\zeta^-} \in B^{s^f}]})
\]

\( \mathbb{P} \)

\[
= \mathbb{P}^x \int_0^\infty f(X_t) 1_{B^{s^f}}(X_t) N(X_t, 1 - \varphi) dH_t + \int_0^\infty f(X_t) 1_{B^{s^f}}(X_t) dK_t
= U(1_{B^{s^f}} f \cdot [N(1 - \varphi) u_H + \nu_K])(x)
\]
where $J$ is the set of jump times for $X$, $(N, H)$ is a Lévy system for $J$, the PCAF $K$ is the dual predictable projection of $1_{[X_{t} - \in \mathcal{E}]} \epsilon_t$, and $\nu_{H}$ and $\nu_{K}$ are the Revuz measures of $H$ and $K$ respectively. By the uniqueness of charges

$$
\pi(dy) = h(y) \nu_{H}(dy) + \nu_{K}(dy) \quad \text{on } B^{\oplus J},
$$

where

$$
h(y) = N(1 - \varphi) = \int_{\mathcal{E}} [1 - \varphi(z)] N(y, dz).
$$

In the context of [1] (symmetric Lévy processes on $\mathbb{R}$), one can take $H_t = t$; moreover $K_t = Ct$ for some constant $C \in [0, \infty]$. Thus, in this case, $\pi$ admits a density with respect to $m$:

$$
\frac{\pi(dy)}{m(dy)} = h(y) + C.
$$

Formula (3) also sharpens the main result of [1] in that $B^{\oplus J}$ (the fine interior of $B$) appears in place of the interior of $B$.

References


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