A NOTE ON "A NOTE ON THE EQUILIBRIUM POTENTIAL OF CERTAIN DIRICHLET SPACES"

P. J. Fitzsimmons

University of California, San Diego

We give a probabilistic proof of the main result of [1]. Let $X = (X_t, \mathbf{P}^x)$ be a symmetric Markov process. We employ the usual notation.

Let B be a Borel subset of E with finite capacity, and let π denote the equilibrium measure for B. Thus, the hitting probability $\varphi(x) := \mathbf{P}^x(T_B < \infty)$ $(x \in E)$ is an excessive m-version of the equilibrium potential $U(\pi)$. In fact,

$$\mathbf{P}^x(f(X_{L_B-}); L_B > 0) = U(f \cdot \pi)(x) \tag{1}$$

for q.e. $x \in E$, for each positive Borel function f. Here L_B is the last exit time from B, so that $\{L_B > 0\} = \{T_B < \infty\}$.

Lemma. *Let $G \in \mathcal{E}^e$ be finely open, and define $g(x) := \mathbf{E}^x(\exp(-T_{G^c}))$. Then g is finely continuous, and*

$$\{t \geq 0 : X_t \in G\} = \{t \geq 0 : g(X_t) < 1\}$$

$\mathbf{P}^x$-a.s. for q.e. $x \in E$.

Proof. The function g is 1-excessive, hence finely continuous. Evidently $G \subset \{g < 1\}$. Moreover, $\{g < 1\} \setminus G$ is the set of points of G^c that are irregular for G^c. This set is semipolar, hence m-polar. ∎

Suppose that f in (1) vanishes outside the fine interior B^{of} of B. Let g be the function in the Lemma corresponding to $G = B^{of}$. If $X_{L_B-} \in B^{of}$ and $X_{L_B-} = X_{L_B}$ then $g(X_{L_B}) < 1$, so $g(X) < 1$ on $[L_B, L_B + \epsilon[$ for some (random) $\epsilon > 0$, because $t \mapsto g(X_t)$ is almost surely right continuous. But this implies that X is in $B^{of} \subset B$ after the last exit time from B, which is absurd. It follows that if $X_{L_B-} \in B^{of}$ then $X_{L_B} \notin B^{of}$; in particular, L_B is a jump time of X on $\{X_{L_B-} \in B^{of}\}$. Thus, for q.e. $x \in E$,

$$\begin{aligned}
&\mathbf{P}^x(f(X_{L_B-}); L_B > 0) \\
&\quad = \mathbf{P}^x \sum_{t \in J, t < \zeta} f(X_{t-}) 1_{\{X_{t-} \in B^{of}\}} 1_{\{T_B \circ \theta_t = \infty\}} + \mathbf{P}^x(f(X_{\zeta-}) 1_{\{X_{\zeta-} \in B^{of}\}}) \\
&\quad = \mathbf{P}^x \int_0^\infty f(X_t) 1_{B^{of}}(X_t) N(X_t, 1-\varphi)\, dH_t + \int_0^\infty f(X_t) 1_{B^{of}}(X_t)\, dK_t \\
&\quad = U(1_{B^{of}} f \cdot [N(1-\varphi)\nu_H + \nu_K])(x)
\end{aligned} \tag{2}$$

where J is the set of jump times for X, (N, H) is a Lévy system for J, the PCAF K is the dual predictable projection of $1_{\{X_{\zeta-} \in E\}} \epsilon_\zeta$, and ν_H and ν_K are the Revuz measures of H and K respectively. By the uniqueness of charges

$$\pi(dy) = h(y)\, \nu_H(dy) + \nu_K(dy) \qquad \text{on } B^{of}, \tag{3}$$

where

$$h(y) = N(1 - \varphi) = \int_E [1 - \varphi(z)]\, N(y, dz).$$

In the context of [1] (symmetric Lévy processes on $\mathbf{R}$), one can take $H_t = t$; moreover $K_t = Ct$ for some constant $C \in [0, \infty[$. Thus, in this case, π admits a density with respect to m:

$$\frac{\pi(dy)}{m(dy)} = h(y) + C.$$

Formula (3) also sharpens the main result of [1] in that B^{of} (the fine interior of B) appears in place of the interior of B.

References

[1] A. Dunkels, *A note on the equilibrium potential of certain Dirichlet spaces*, Potenital Analysis **6** (1997), 99–104.

Department of Mathematics, University of California San Diego, 9500 Gilman Drive, La Jolla, CA 92093–0112

E-mail address: `pfitz@euclid.ucsd.edu`