1. Introduction

Let $X = (X_t)_{t \geq 0}$ be a regular diffusion process on an interval $E \subset \mathbb{R}$. Let $H_t := \min_{0 \leq u \leq t} X_u$ denote the past minimum process of $X$ and consider the excursions of $X$ above its past minimum level: If $[a, b]$ is a maximal interval of constancy of $t \mapsto H_t$, then $(X_t : a \leq t \leq b)$ is the “excursion above the minimum” starting at time $a$ and level $y = H_a$. These excursions, when indexed by the level at which they begin, can be regarded (collectively) as a point process. The independent increments property of the first-passage process of $X$ implies that this point process is Poissonian in nature, albeit non-homogeneous in intensity. Moreover, intuition tells us that the distribution of an excursion above the minimum $(X_t : a \leq t \leq b)$ should be governed by the Itô excursion law corresponding to excursions above the fixed level $y = H_a(\omega)$.

Our first task is to render precise the ruminations of the preceding paragraph. This is accomplished in sections 2 and 3 by applying Maisonneuve’s theory of exit systems [10] to a suitable auxiliary process $(\tilde{X}_t)$ associated with $X$. The basic result, stated in section 2, affirms the existence of a “Lévy system” for the point process of excursions of $X$ above its past minimum.

In sections 4, 5, and 6 we discuss several applications of the Lévy system constructed in section 3; these applications concern path decompositions of $X$ involving the minimum process $H$. Such decompositions, and related results, have been found by various authors (see [6, 9, 11, 12, 14, 15, 16, 17, 18]), most often in the special case where $X$ is Brownian motion. The possibility of using Lévy systems to give a unified treatment of path decompositions is, of course, not surprising. In an excellent synthesis [13] Pitman has shown how the existence of a Lévy system for a point process attached to a Markov process leads naturally to various path decompositions of the Markov process.

In section 4 we obtain a general version of Williams’ decomposition of a diffusion at its global minimum. A “local” version of Williams’ decomposition can be found in section 5. In section 6 we give a new proof of a result of Vervaat [17], which states that a Brownian bridge, when split at its minimum and suitably “rearranged” becomes a (scaled) Brownian
excursion. Indeed, we produce an inversion of Vervaat’s transformation, showing how a Brownian excursion may be split and rearranged to yield Brownian bridge.

2. Notation and the basic result

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x)$ be a canonically defined regular diffusion on an interval $E \subset \mathbb{R}$. Here $\Omega$ denotes the space of paths $\omega: [0, +\infty] \to E \cup \{\Delta\}$ which are absorbed in the cemetery point $\Delta \notin E$ at time $\zeta(\omega)$, and which are continuous on $[0, \zeta(\omega)]$. For $t \geq 0$, $X_t(\omega) = \omega(t)$, and $\theta_t \omega$ denotes the path $u \mapsto \omega(u + t)$. The $\sigma$-fields $\mathcal{F}$ and $\mathcal{F}_t$ ($t \geq 0$) are the usual Markovian completions of $\mathcal{F} = \sigma\{X_u: u \geq 0\}$ and $\mathcal{F}_t = \sigma\{X_u: 0 \leq u \leq t\}$ respectively. The law $P^x$ on $(\Omega, \mathcal{F}^\circ)$ corresponds to $X$ started at $x \in E$. We shall also make use of the killing operators $(k_t)$ defined for $t \geq 0$ by

$$k_t \omega(u) = \begin{cases} \omega(u), & u < t, \\ \Delta, & u \geq t. \end{cases}$$

Let $A = \inf E$, $B = \sup E$, and write $E^\circ = ]A, B[$. We assume throughout the paper that $A \notin E$, and that $B \in E$ if and only if $B$ is a regular boundary point which is not a trap for $X$. In particular, these assumptions imply that the transition kernels of $X$ are absolutely continuous with respect to the speed measure $m$ (recalled below). See §4.11 of Itô-McKean [8].

Let $s$ (resp. $m$, resp. $k$) denote a scale function (resp. speed measure, resp. killing measure) for $X$. Recall from [8] that the generator $\mathcal{G}$ of $X$ has the form

$$\mathcal{G}u(x) \cdot m(dx) = du^+(x) - u(x) \cdot k(dx), \quad x \in E^\circ,$$

for $u \in D(\mathcal{G})$, the domain of $\mathcal{G}$. Here and elsewhere $u^+$ denotes the scale derivative:

$$u^+(x) = \lim_{y \downarrow x} \frac{u(y) - u(x)}{s(y) - s(x)}.$$

Let $(U^\alpha : \alpha > 0)$ denote the resolvent family of $X$. Subsequent calculations require an explicit expression for the density of $U^\alpha(x, dy)$ with respect to $m(dy)$. Recall from [8] that for each $\alpha > 0$ there are strictly positive, linearly independent solutions $g_{1\alpha}$ and $g_{2\alpha}$ of

$$\mathcal{G}g(x) = \alpha g(x), \quad x \in E^\circ;$$

$g_{1\alpha}$ (resp. $g_{2\alpha}$) is an increasing (resp. decreasing) solution of (2.2) which also satisfies the appropriate boundary condition at $A$ (resp. $B$). Both $g_{1\alpha}$ and $g_{2\alpha}$ are uniquely determined
up to a positive multiple. We sometimes drop the superscript $\alpha$, writing simply $g_1$ and $g_2$. Since $g_1$ and $g_2$ are linearly independent solutions of (2.2), the Wronskian $W = g_1^+ g_2 - g_2^+ g_1$ is constant. The resolvent $U^\alpha$ is given by
\begin{equation}
U^\alpha f(x) = U^\alpha(x, f) = \int_E u^\alpha(x, y) f(y) \, m(dy),
\end{equation}
where
\begin{equation}
u^\alpha(x, y) = u^\alpha(y, x) = g_1^\alpha(x) g_2^\alpha(y) / W, \quad x \leq y.
\end{equation}
See §4.11 of [8] and note that in (2.3) the mass $m(\{B\})$ is the “stickiness” coefficient occurring in the boundary condition at $B$ for elements of $D(G)$.

A jointly continuous version $(L^y_t : t \geq 0, y \in E)$ of local time for $X$ may be chosen, and normalized to be occupation density relative to $m$, so that
\begin{equation}
P^x \int_0^\infty e^{-\alpha t} \, dL^y_t = u^\alpha(x, y).
\end{equation}
Fixing a level $y \in E$, the local time $(L^y_t : t \geq 0)$ is related to the Itô excursion law [7], for excursions from level $y$, as follows. Let $G(y)$ denote the (random) set of left-hand endpoints (in $]0, \zeta[)$ of intervals contiguous to the level set $\{t > 0 : X_t = y\}$. Define the hitting time $T_y$ by
\[T_y = \inf\{t > 0 : X_t = y\} \quad (\inf \emptyset = +\infty).\]
The Itô excursion law $n_y$ is determined by the identity
\begin{equation}
P^x \sum_{u \in G(y)} Z_u F \circ k_{T_y} \circ \theta_u = P^x \left( \int_0^\infty Z_u \, dL^y_u \right) \cdot n_y(F),
\end{equation}
where $x \in E$, $F \in pF^\circ$, and $Z \geq 0$ is an $(\mathcal{F}_t)$-optional process. Under $n_y$ the coordinate process $(X_t : t > 0)$ is strongly Markovian with semigroup $(Q_t^y)$ given by
\begin{equation}Q_t^y(x, f) = P^x(f \circ X_t; t < T_y).
\end{equation}
The entrance law $n_y(X_t \in dz)$ is determined by the corresponding Laplace transform
\begin{equation}W^\alpha f(y) = W^\alpha(y, f) = n_y \int_0^\zeta e^{-\alpha t} f \circ X_t \, dt.
\end{equation}
Conversely, $n_y$ is the unique $\sigma$-finite measure on $(\Omega, \mathcal{F}^\circ)$ which is carried by $\{\zeta > 0\}$ and under which $(X_t : t > 0)$ is Markovian with semigroup (2.7) and entrance law (2.8).
Let $V_y^\alpha$ denote the resolvent of the semigroup $(Q_y^t)$. Taking $Z_u = e^{-\alpha u}$, $F = \int_0^\zeta e^{-\alpha t} f(X_t) dt$ in (2.6), and using (2.5), we obtain the important identity

$$U^\alpha f(x) = V_y^\alpha f(x) + u^\alpha(x, y)[m(\{y\})f(y) + W^\alpha f(y)].$$

We also recall from §4.6 of [8] that the distribution of $T_y$ is given by

$$P^x(e^{-\alpha T_y}) = \begin{cases} \frac{g_1^\alpha(x)}{g_1^\alpha(y)}, & x \leq y, \\ \frac{g_2^\alpha(x)}{g_2^\alpha(y)}, & x \geq y. \end{cases}$$

Finally, the point process of excursions above the minimum is defined as follows. For $t \geq 0$ set

$$H_t(\omega) = \begin{cases} \min_{0 \leq u \leq t} X_u(\omega) & \text{if } t < \zeta(\omega), \\ -\infty & \text{if } t \geq \zeta(\omega); \end{cases}$$

$$M(\omega) = \{u > 0 : X_u(\omega) = H_u(\omega)\};$$

$$R_t(\omega) = \inf\{u > 0 : u + t \in M(\omega)\};$$

$$G(\omega) = \{u > 0 : u < \zeta(\omega), R_u(\omega) = 0 < R_u(\omega)\}.$$  

Thus $G$ is the random set of left-hand endpoints of intervals contiguous to the random set $M$. For $u \in G$ we have the excursion $e^u$ defined by

$$e^u_t = \begin{cases} X_{u+t}, & 0 \leq t < R_u, \\ \Delta, & t \geq R_u. \end{cases}$$

The point process $\Pi = (e^u : u \in G)$ admits a Lévy system as follows. Define a continuous increasing adapted process $C = (C_t : t \geq 0)$ by

$$C_t = \begin{cases} s(H_0) - s(H_t), & \text{if } t < \zeta, \\ C_\zeta, & \text{if } t \geq \zeta. \end{cases}$$

(2.11) Theorem. For $Z \geq 0$ and $(\mathcal{F}_t)$-optional, and $F \in \mathcal{P}\mathcal{F}^\circ,$

$$P^x \sum_{u \in G} Z_u F(e^u) = P^x \int_0^\infty Z_u n_{X_u}^\uparrow (F) dC_u$$

$$= P^x \int_x^\infty Z_{T_y} 1_{\{T_y < +\infty\}} n_y^\uparrow (F) ds(y),$$

where $n_y^\uparrow$ denotes the restriction of $n_y$ to $\{\omega : \omega(t) > y, \forall t \in ]0, \zeta(\omega)[\}.$

(2.13) Remark. The second equality in (2.12) follows from the first by the change of variable $u = T_y$. The equality of the first and third terms in (2.12) amounts to the
statement that the time-changed point process \((e_T^y : R_{T^-} < R_T^y, A < y < x)\) is a stopped Poisson point process under \(P^x\), with (non-homogeneous) intensity \(ds(y)n^1_y(d\omega)\), stopped at the first level \(y\) for which \(T_y = +\infty\). See [14] for this result in the case of Brownian motion, with or without drift. The general result (2.11) was suggested by §4.10 of [8].

3. Proof of Theorem (2.11)

Maisonneuve’s theory of exit systems [10] provides a Lévy system description of the point process of excursions induced by a closed, optional, homogeneous random set. Unfortunately the set \(M\) introduced in §2 is not \((\theta_t)\)-homogeneous; however the theory of [10] can be brought to bear once we note that \(M\) is homogeneous as a functional of the strong Markov process \((X_t, H_t)\), \(t \geq 0\). This key observation is due to Millar [12] and has been formalized by Getoor in [4]. In the terminology of [4], the process \(H\) is a “min-functional”: \(H_t + u = H_t \wedge H_u \circ \theta_t\). This property ensures that \(\overline{X} := (X, H)\) is Markovian, as a simple computation shows.

Following [4] we first construct a convenient realization of \(\overline{X}\). Let \(\overline{\Omega} = \Omega \times (E \cup \{-\infty\})\), \(E = \{(x,a) \in E \times E : a \leq x\}\), and for \((\omega,a) \in \overline{\Omega}\) set
\[
\overline{X}_t(\omega, a) = (X_t(\omega), a \wedge H_t(\omega)),
\]
\[
\overline{\theta}_t(\omega, a) = (\theta_t(\omega), a \wedge H_t(\omega)).
\]
Clearly \(\overline{X}_t \circ \overline{\theta}_u = \overline{X}_{t+u}\), \(\overline{\theta}_t \circ \overline{\theta}_u = \overline{\theta}_{t+u}\). Moreover, \(M\) can be realized over \(\overline{X}\) as
\[
(3.1) \quad \overline{M}(\omega, a) = \{t > 0 : \overline{X}_t(\omega, a) \in D\},
\]
where \(D = \{(x,x) : x \in E\}\). Let \(\overline{\mathcal{F}}^\circ = \sigma\{\overline{X}_u : u \geq 0\}\), \(\overline{\mathcal{F}}_t^\circ = \sigma\{\overline{X}_u : 0 \leq u \leq t\}\), and for \((x,a) \in E\) let \(P^{x,a} = P^x \otimes \epsilon_a\). The usual Markovian completion of the filtration \((\overline{\mathcal{F}}_t^\circ)\) relative to the laws \((P^{x,a} : (x,a) \in E)\) is denoted by \((\overline{\mathcal{F}}_t)\). Clearly \(P^{\overline{X}_0, a}(X_0 = (x,a)) = 1\) so that \(\overline{X}\) has no branch points. Appealing to §2 of [4] we have the following

(3.2) Lemma.  (i) \(\overline{X} = (\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{\theta}_t, \overline{X}_t, P^{x,a})\) is a right-continuous, strong Markov process with state space \(\overline{E}\) and cemetery \(\overline{\Delta} = (\Delta, -\infty)\). The semigroup of \(\overline{X}\) maps Borel functions to Borel functions, so that \(\overline{X}\) is even a Borel right process.

(ii) Let \(\pi: (\omega,a) \rightarrow \omega\) denote the projection of \(\overline{\Omega}\) onto \(\Omega\). If \(Z\) is an \((\mathcal{F}_t)\)-optional process, then \(Z \circ \pi\) is \((\mathcal{F}_t)\)-optional.

Now \(\overline{M}\) is an \((\overline{\mathcal{F}}_t)\)-optional, \((\overline{\theta}_t)\)-homogeneous set, and each section \(\overline{M}(\omega, a)\) is closed in \(]0, \overline{\zeta}(\omega, a)\]. Set \(\overline{R} = \inf \overline{M}\), so that \(\overline{R}\) is an exact terminal time of \(\overline{X}\) with \(\text{reg}(\overline{R}) = \overline{R}\).
\((x, a) \in \overline{E} : \overline{\mathcal{P}}^{x,a}(R = 0) = 1\} = D\). This last fact follows from the regularity of \(X\) and the identity
\[
\overline{\mathcal{P}}^{x,a}(R = T_a \circ \pi) = 1, \quad (x, a) \in \overline{E}.
\]
Let \(\overline{G}\) denote the set of left-hand endpoints of intervals contiguous to \(\overline{M}\). The properties of the Maisonneuve exit system \((^*\overline{\mathcal{P}}^{x,a}, \overline{K})\) for \(\overline{M}\) are summarized in the next proposition.

In what follows, \(\mathcal{E}^*\) and \(\mathcal{F}^*\) denote the universal completions of \(\mathcal{E}\) (the Borel sets in \(\mathcal{E}\)) and \(\mathcal{F}\) respectively.

**Proposition.** [Maisonneuve] There exists a continuous additive functional (CAF), \(K\), of \(X\) with a finite 1-potential, and a kernel \(^\star \mathcal{P}^{x,a}\) from \((\mathcal{E}, \mathcal{E}^*)\) to \((\Omega, \mathcal{F}^*)\) such that
\[
\mathcal{P}^{x,a} \sum_{u \in \overline{G}} Z_u F_u \circ \theta_u = \mathcal{P}^{x,a} \int_0^\infty Z_u \mathcal{P}^X_u(F_u) \, dK_u,
\]
whenever \(Z \geq 0\) is \((\mathcal{F}_t)\)-optional and \((u, \omega) \mapsto F_u(\omega)\) is a \(\mathcal{B}_{[0, +\infty[} \otimes \mathcal{F}^*\)-measurable, positive function. The CAF \(K\) is carried by \(D\). For each \((x, a) \in \mathcal{E}\), \(^* \mathcal{P}^{x,a}\) is a \(\sigma\)-finite measure on \((\Omega, \mathcal{F}^*)\) under which the coordinate process is strongly Markovian with the same transition semigroup as \(\overline{X}\).

**Remarks.** The version of \((^* \overline{\mathcal{P}}^{x,a}, \overline{K})\) cited in (3.3) is a variant of that constructed in [10]; the difference stems from the possibility that \(\overline{\mathcal{P}}^{x,a}(\zeta < +\infty)\) may be positive. The fact that \(K\) is continuous (and so carried by \(D = \operatorname{reg}(\overline{R})\)) follows from the construction in [10], since \(\overline{M} = \{t > 0 : \overline{X}_t \in D\}\) and \(D\) is finely perfect (with respect to \(\overline{X}\)). Renormalizing the kernel \(^* \overline{\mathcal{P}}^{x,a}\) if necessary, we can and do assume that \(^* \mathcal{P}^{y,y}(1 - e^{-\overline{R}}) = 1\) for all \(y \in E\).

Our plan is to prove Theorem (2.11) by identifying \(^* \overline{\mathcal{P}}^{x,a}\) and \(\overline{K}\) explicitly, thereby deducing (2.12) from (3.4). First note that by taking \(x = y\) in (2.9) and using (2.4) we have
\[
W^\alpha f(y) = \int_{[A,y]} [g_1^\alpha(z)/g_1^\alpha(y)] f(z) \, m(dz) + \int_{[y,B]} [g_2^\alpha(z)/g_2^\alpha(y)] f(z) \, m(dz),
\]
where \(y \in E\), \(\alpha > 0\), and \(f \geq 0\) is Borel measurable on \(E\).

To identify \(\overline{K}\) we define a second CAF of \(\overline{X}\), \(\overline{C}\), by the formula
\[
\overline{C}_t(\omega, a) = \begin{cases} s(a \wedge H_0(\omega)) - s(a \wedge H_t(\omega)) & \text{if } t < \overline{\zeta}(\omega, a), \\ \overline{C}_{\overline{\zeta}_-}(\omega, a) & \text{if } t \geq \overline{\zeta}(\omega, a); \end{cases}
\]
and notice that \(\overline{C}_t(\omega, X_0(\omega)) = C_t(\omega)\). Clearly the fine support of \(\overline{C}\) is \(D\).

For \(x \in E\) put \(\psi(x) = W 1_{[x,B]}(x)\).
Proposition. The CAFs $\overline{K}$ and $\int_0^t \psi(X_u) dC_u$ are equivalent.

Proof. By [1; IV(2.13)] it suffices to check that the CAFs in question have the same finite 1-potential (over $\overline{X}$). An argument of Vervaat [17] shows that $P^x(t \in M) = 0$ for all $x \in E$ and all $t > 0$. Consequently, $\overline{P}^{x,a}(t \in \overline{M}) = 0$ for all $(x, a) \in \overline{E}$ and all $t > 0$. By Fubini’s theorem, $\int_0^\infty e^{-t} \overline{M}(t) dt = 0$ a.s. $\overline{P}^{x,a}$ for all $(x, a) \in \overline{E}$. Thus taking $\overline{Z}_u(\omega) = e^{-u}$, $\overline{F}_u(\omega) = 1 - \exp(-\overline{R}(\omega) \wedge \zeta(\omega))$ in (3.4), we may compute

$$
\overline{P}^{x,a} \int_0^\infty e^{-u} d\overline{K}_u = \overline{P}^{x,a} \sum_{u \in G} e^{-u} \left( \int_0^{\overline{R} \wedge \zeta} e^{-t} \right) d\overline{\theta}_u.
$$

(3.8)

where the last equality follows easily from (2.9). On the other hand, our hypothesis regarding the boundary $A$ implies that $g_1^1(A+/g_2^1(A+) = 0$ (see [8; §4.6]). Thus

$$
\overline{P}^{x,a} \int_0^\infty e^{-t} \psi(X_t) d\overline{C}_t = \overline{P}^{x} \int_{T_a}^\infty e^{-t} \psi(X_t) dC_t
$$

(3.9)

$$
= \overline{P}^{x} \int_A^a e^{-T_x \psi(y)} ds(y).
$$

In (3.9) we have used the change of variables $t = T_y$ to obtain the second equality, and (2.1) to obtain the third. Now from the definition of the Wronskian $W$ we see that $d(g_1/g_2) = W \cdot [g_2]^{-2} ds$. Using this fact and the expression for $\psi$ provided by (3.6) we
may continue the computation begun in (3.9) with

\[
\begin{align*}
= \int_A \left[ g_2(x)/g_2(y) \right] \int_{[y,B]} \left[ g_2(z)/g_2(y) \right] m(dz) \, ds(y) \\
= \int_A \int_{[y,B]} \left[ g_2(x)g_2(z)/W \right] m(dz) \, d(g_1/g_2)(y) \\
= \int_E \int_A \int_{[y,B]} \left[ g_1(z \wedge a)/g_2(z \wedge a) \right] \left[ g_2(x)g_2(z)/W \right] m(dz) \\
= \left[ u^1(x,a)/u^1(a,a) \right] U^1 1(a).
\end{align*}
\]

(3.10)

The last equality in (3.10) follows from (2.3) and (2.4). In view of (3.8)–(3.10), we see that \( K \) and \( \int_0^t \psi(X_s) \, dC_s \) have the same finite 1-potential and so the proposition is proved. \( \square \)

For \( y \in E \) define a measure \( \overline{Q}^y \) on \((\Omega, \mathcal{F})\) by \( \overline{Q}^y(F) = *T^{y,F}(F \circ k_r) \), where \( k_t \) is the killing operator on \( \Omega \). Since \( *T^{y,F}(F \neq T_y \circ \pi) = 0 \), the first coordinate of \( X \), namely \( (X_t : t > 0) \), is Markovian under \( \overline{Q}^y \), with \( (Q_t^y) \) as semigroup. Indeed, we claim that \( \psi(y) \pi(\overline{Q}^y) = n^1_y \), at least for \( ds \)-a.e. \( y \in E \). To verify this claim it suffices to compare the associated entrance laws.

(3.11) Lemma. Let \( f \) be a bounded positive Borel function on \( E \). Then for \( ds \)-a.e. \( y \in E \) we have

\[
\psi(y) \overline{Q}^y \int_0^\infty e^{-\alpha t} f(X_t) \, dt = W^\alpha f(y), \quad \forall \alpha > 0.
\]

(3.12)

Proof. Fix \( f \) as in the statement of the lemma and also fix \( \alpha > 0 \). For \( y \in E \) write

\[
\gamma(y) = \overline{Q}^y \int_0^\infty e^{-\alpha t} f(X_t) \, dt.
\]

As noted in the proof of (3.7), we have \( \int_0^\infty 1_M(t) \, dt = 0 \), \( P^{x,a} \)-a.s. for all \((x,a) \in E\). Thus,
for \( x \in E \),

\[
U^\alpha f(x) = P^{x,x} \sum_{u \in G} e^{-\alpha u} \left( \int_0^\infty e^{-\alpha t} f(X_t) \, dt \right) \circ \theta_u
\]

\[
= P^{x,x} \int_0^\infty e^{-\alpha u} P^X_u \left( \int_0^\infty e^{-\alpha t} f(X_t) \, dt \right) \, dK_u
\]

\[= P^{x,x} \int_0^\infty e^{-\alpha u} \gamma(X_u) \psi(X_u) \, dC_u
\]

\[= P^x \int_A e^{-\alpha T(y)} \gamma(y) \psi(y) \, ds(y)
\]

\[= \int_A [g^\alpha_1(x)/g^\alpha_2(y)] \gamma(y) \psi(y) \, ds(y).
\]

(3.13)

On the other hand, by (2.3) and (2.4), we have

(3.14) \( U^\alpha f(x) = \int_A \left[ g^\alpha_1(x)/g^\alpha_2(y) \right] f(y) \, m(dy) + \int_{[x, B]} \left[ g^\alpha_1(y)/g^\alpha_2(x) \right] f(y) \, m(dy). \)

If we equate the last line displayed in (3.13) with the right side of (3.14), divide the resulting identity by \( g^\alpha_2(x) \), and then differentiate in \( x \), we obtain

\[
\int_{[x, B]} \left[ g^\alpha_1(y)/g^\alpha_2(x) \right] f(y) \, m(dy) \, ds(x) = \gamma(x) \psi(x) \, ds(x)
\]

as measures on \( E \), and the lemmas follows. 

(3.15) Corollary. For \( ds\text{-a.e. } y \in E \), \( \psi(y) \pi(Q^y) = n^+_y \) as measures on \( (\Omega, \mathcal{F}^\circ) \).

Proof. As noted earlier, both \( \psi(y) \pi(Q^y) \) and \( n^+_y \) make the coordinate process on \( (\Omega, \mathcal{F}^\circ) \) into a Markov process with transition semigroup \( (Q^y_t) \). It follows from Lemma (3.11) that these measures have the same one-dimensional distributions (and consequently the same finite dimensional distributions) for \( ds\text{-a.e. } y \). Since \( \mathcal{F}^\circ = \sigma(X_u : u \geq 0) \) is countably generated, the corollary follows. 

Proof of Theorem (2.11). Let \( Z \geq 0 \) be \( (\mathcal{F}_t) \)-optional and let \( F \geq 0 \) be \( \mathcal{F}^\circ \)-measurable. By (3.2)(ii), the process \( Z \circ \pi \) is \( (\mathcal{F}_t) \)-optional. We may now use (3.4), (3.7), and (3.15) to
compute
\[ P^x \sum_{u \in G} Z_u F(e^u) = P^{x,x} \sum_{u \in G} Z_u \circ \pi F(\pi \circ k \circ \theta_u) \]
\[ = P^{x,x} \int_0^\infty Z_u \circ \pi \mathbb{X}_u \left(F \circ \pi \circ k \right) dK_u \]
\[ = P^{x,x} \int_0^\infty Z_u \circ \pi \mathbb{X}_u \left(F \circ \pi \circ k \right) \psi(X_u) dC_u \]
\[ = P^x \int_0^\infty Z_u \circ \pi \mathbb{X}_u \left(F \circ \pi \circ k \right) dC_u. \]
The proof of Theorem (2.11) is complete. \( \square \)

4. Williams’ decomposition

In this section we use the Lévy system (2.12) to obtain a new proof (of a general version) of Williams’ decomposition [18] of a diffusion at its global minimum. A more “computational” proof of Williams’ theorem, based on the same idea used in the present paper, may be found in [3].

For simplicity we assume that \( \gamma := H_{\zeta_-} \) satisfies \( P^x(\gamma > A) = 1 \) for all \( x \in E \). We also assume that \( \rho := \inf \{ t > 0 : X_t = \gamma \} \) satisfies \( P^x(\rho < \zeta) = 1 \) for all \( x \in E \). Then \( \rho \) is the unique time at which \( X \) takes its global minimum value \( \gamma \) (cf. [17]). Note that for \( x \geq y \) (both in \( E \)),
\[ P^x(\gamma > y) = P^x(T_y = +\infty). \] (4.1)

Define a function \( r \) on \( E \) by
\[ r(x) = \begin{cases} P^x(T_{x_0} < +\infty), & x \geq x_0, \\ \left[P^{x_0}(T_x < +\infty)\right]^{-1}, & x < x_0, \end{cases} \]
where \( x_0 \in E^\circ \) is fixed but arbitrary. Clearly \( r \) is strictly positive and decreasing. Arguing as in [8; pp. 128–129] one may check that \( r \) is the unique positive decreasing solution of \( Gr \equiv 0 \) on \( E^\circ \) which satisfies \( r(x_0) = 1 \) and the boundary condition at \( B \). Note that
\[ P^x(T_y < +\infty) = r(x)/r(y), \quad x > y. \] (4.2)

Before proceeding to the decomposition theorem we need a preliminary result.

(4.3) Lemma. For \( y \in E^\circ \) let \( S_y = \inf \{ t > 0 : X_{t-} = y \} \). Then
\[ n_y^\dagger(S_y = +\infty) = -\frac{r^+(y)}{r(y)}, \quad \forall y \in E^\circ. \]
(Recall that \( r^+ = d^+ r / ds^+ \).)

**Proof.** Let \( q \) be an increasing solution of \( Gq \equiv 0 \) on \( E^\circ \) such that \( q \) is linearly independent of \( r \). We assume that \( q \) is normalized so that the Wronskian \( q^+ r - r^+ q \) is identically 1. Fix \( a < b \) both in \( E^\circ \) and let \( v_{ab} \) denote the potential density (relative to \( m \)) of \( X \) killed at time \( T_a \land T_b \). One checks that for \( x \leq y \),

\[
v_{ab}(x, y) = v_{ab}(y, x) = \frac{D(a, x)D(y, b)}{D(a, b)},
\]

where \( D(x, y) \) is the determinant

\[
\begin{vmatrix}
q(y) & q(x) \\
r(y) & r(x)
\end{vmatrix}.
\]

Note that \( D(x, y) > 0 \) if \( x < y \). Now let \( y \in [a, b[ \) and use (2.6) to compute

\[
P^y(T_b < T_a) = P^y \sum_{u \in G(y)} 1_{\{u < T_a \land T_b\}} 1_{\{T_b < T_y\}} \cdot \theta_u
\]

\[
= P^y(L^y_{T_a \land T_b}) n_y(T_b < \zeta).
\]

But clearly \( P^y(T_b < T_a) = D(a, y) / D(a, b) \) while \( P^y(L^y_{T_a \land T_b}) = v_{ab}(y, y) \), so that

\[
n_y(T_b < \zeta) = [D(a, y) / D(a, b)] / v_{ab}(y, y) = D(y, b)^{-1}.
\]

Finally,

\[
n^+_y(S_y = +\infty) = \lim_{x \downarrow y} n^+_y(T_x < +\infty, S_y = +\infty)
\]

\[
= \lim_{x \downarrow y} n^+_y(T_x < +\infty) P^x(T_y = +\infty)
\]

\[
= \lim_{x \downarrow y} n_y(T_x < \zeta) [1 - r(x) / r(y)]
\]

\[
= \lim_{x \downarrow y} \left[ \frac{1}{r(y)} \cdot \frac{r(y) - r(x)}{s(x) - s(y)} \cdot \frac{s(x) - s(y)}{D(y, x)} \right]
\]

\[
= -r(y)^{-1} \cdot r^+(y),
\]

since \( \lim_{x \downarrow y} [s(x) - s(y)] / D(y, x) \) is the reciprocal of the Wronskian \( q^+ r - r^+ q \equiv 1 \). □

Now define probability laws on \( (\Omega, \mathcal{F}^\circ) \) by

\[
(4.4) \quad P^x(F) = P^x(F \circ k_{T_y} | T_y < +\infty),
\]

\[
(4.5) \quad P^+_y(F) = n^+_y(F | S_y = +\infty),
\]
whenever \( x > y > A \). The coordinate process is a diffusion under any of these laws: \( P^x_y \) is the law of \( X \) started at \( x \), conditioned to converge to \( A \), and then killed at \( T_y \); \( P^y_y \) is the law of \( X \) started at \( y \) and conditioned to never return to \( y \). These conditionings are accomplished by means of the appropriate \( h \)-transforms. In particular, the associated infinitesimal generators are given by

\[
G^x_y f(z) = r(z)^{-1}G(fr)(z), \quad z > y;
\]

\[
G^y_y f(z) = r_y(z)^{-1}G(fr_y)(z), \quad z > y,
\]

where \( r_y(z) = 1 - r(z)/r(y) \).

We can now state the general version of Williams’ theorem. Recall that \( \gamma = H_\zeta - \) and \( \rho = \inf\{t > 0 : X_t = \gamma \} \).

**Theorem.** (a) The joint law of \((\gamma, \rho, \zeta)\) is given by

\[
P^x(f) e^{-\alpha \rho - \beta \zeta} = \int_A \left[ g^{\alpha + \beta}(x)/g^{\alpha + \beta}(y) \right] f(y) P^y_1(e^{-\beta \zeta}) \frac{-dr(y)}{r(y)}.
\]

(b) For \( F,G \in b\mathcal{F}^\circ \) and \( \psi \) bounded and Borel on \( E \),

\[
P^x(F \circ k_\rho \psi(\gamma) G \circ \theta_\rho) = P^x(P^x_y(F \psi(\gamma) P^y_1(G)).
\]

**Remark.** The intuitive content of (4.10) is that the processes \((X_t : 0 \leq t < \rho)\) and \((X_{\rho+t} : 0 \leq t < \zeta - \rho)\) are conditionally independent under \( P^x \), given \( \gamma \); and that the conditional distributions, given that \( \gamma = y \), are \( P^x_y \) and \( P^y_y \) respectively.

**Proof of (4.8).** Define \( J(y, \omega) = 1_{\{S_y = +\infty\}}(\omega) \) and observe that \( \rho(\omega) = u \) if and only if \( u \in G(\omega) \) and \( J(X_u(\omega), e^u(\omega)) = 1 \). Thus, using (2.11),

\[
P^x(F \circ k_\rho \psi(\gamma) G \circ \theta_\rho) = P^x \sum_{u \in G} F \circ k_u \psi(X_u) G(e^u) J(X_u, e^u)
\]

\[
= \int_A P^x(F \circ T_y; T_y < +\infty) \psi(y) n^1_y(G \cdot J(y, \cdot)) \, ds(y)
\]

\[
= \int_A P^x_1(F) \psi(y) P^1_y(G) P^x(T_y < +\infty) n^1_y(S_y = +\infty) \, ds(y).
\]

Taking \( F = G = 1 \) in (4.12) we see that

\[
P^x(\gamma \in dy) = P^x(T_y < +\infty) n^1_y(S_y = +\infty) \, ds(y).
\]
Now (4.13) substituted into the last line of (4.12) yields (4.10). To obtain (4.9) use (4.10) with
\[ F = e^{-(\alpha + \beta)\zeta} \] and \( G = e^{-\alpha \zeta} \), noting that \( F \circ k_\rho = e^{-(\alpha + \beta)\rho} \) and \( \zeta = \rho + \zeta \theta_\rho \) (\( P^x \)-a.s.) since \( \rho < \zeta \), \( P^x \)-a.s. Thus
\[
P^x(f(\gamma)e^{-\alpha \rho}e^{-\beta \zeta}) = P^x(f(\gamma)[e^{-(\alpha + \beta)\zeta} \circ k_\rho [e^{-\beta \zeta} \circ \theta_\rho])
\]
\[
= P^x(P_{\gamma}^{x\perp}(e^{-(\alpha + \beta)\zeta})f(\gamma)P_{\gamma}^I(e^{-\beta \zeta})).
\]
Formula (4.9) now follows since
\[
P^x(\rho \in dt, \gamma \in dy) = P^x(T_y \in dt) - dr(y)/r(y).
\]

(4.14) Corollary. \( P^x(\rho \in dt, \gamma \in dy) = P^x(T_y \in dt) \frac{-dr(y)}{r(y)}. \)

5. A local decomposition
Fix \( t > 0 \) and define
\[
\rho_t = \inf\{u > 0 : X_u = H_t\} \land t.
\]
Arguing as in [17] one can show that, almost surely on \( \{t < \zeta\} \), \( \rho_t \) is the unique \( u \in ]0,t[ \) such that \( X_u = H_t \). Our purpose in this section is to describe the conditional distribution of \( \{X_u : 0 \leq u \leq t\} \) under \( P^b \), given that \( H_t = y, \rho_t = u, \) and \( X_t = x \). This conditional distribution has been computed by Imhof [6] for the Brownian motion (and closely related processes). The joint law of \( (H_t, \rho_t, X_t) \), again in the case of Brownian motion, has been found by Shepp [16]. See also [2, 9, 15] for related results.

We begin by computing the joint law of \( (H_t, \rho_t, X_t) \). Recall from [8; §4.11] that the first passage distribution \( P^x(T_y \in dv) \) has a density \( f(v; x, y) \) on \( ]0, +\infty[ \) relative to Lebesgue measure. Note that if we set \( F_{t,y}(x) = P^x(t < T_y < +\infty) \), then (see [8; p. 154])
\[
f(t; x, y) = -\frac{\partial}{\partial t} F_{t,y}(x) = GF_{t,y}(x), \quad x > y \in E, t > 0.
\]
Applying \( Q^y(z, dx) \) to both sides of (5.1) and integrating over \( x \in ]y, +\infty[ \cap E \) (making use of the relation \( Q^y_s \mathcal{G} = \mathcal{G} Q_s^y \) on \( ]y, +\infty[ \)), we obtain
\[
f(t + s; z, y) = \int_{]y, +\infty[} Q^y_s(z, dx) f(t; x, y).
\]
In other words, \((t, x) \mapsto f(t; x, y)\) is an exit law for the semigroup \((Q^y_t)\).

Next, recall from [6; §4.11] that the semigroup \((Q^y_t)\) has a density \(q^y(t; x, z)\) (for \(x \land z > y\)) relative to the speed measure \(m(dz)\); we have \(q^y > 0\) on \([0, +\infty[ \times (|y, B|)^2\) and \(q^y(t; x, z) = q^y(t; z, x)\). The entrance law for \(n^1_y\) can now be expressed as

\[
(5.2) \quad n^1_y(X_t \in dx) = q^1_y(t; x) m(dx),
\]

where

\[
(5.3) \quad q^1_y(t; x) = \int_{[y, B]} n^1_y(X_{t-u} \in dz) q^y(u; z, x).
\]

Substituting (5.2) into (5.3) and using the symmetry of \(q^y\), we see that

\[
(5.4) \quad q^1_y(t + u; x) = \int_{[y, B]} Q^y_u(x, dz) q^1_y(t; z).
\]

But (5.4) means that \((t, x) \mapsto q^1_y(t; x)\) is also an exit law for \((Q^y_t)\). Finally, using (3.6), if \(\alpha > 0\) and \(h\) is positive, measurable, and vanishes off \([y, B]\), we may compute

\[
\int_0^\infty e^{-\alpha t} \int_E q^1_y(t; x) h(x) m(dx) dt = W^\alpha h(y)
\]

\[
= \int_{[y, B]} \left[ g_2^\alpha(x)/g_2^\alpha(y) \right] h(x) m(dx)
\]

\[
= \int_{[y, B]} P^x e^{-\alpha T^y} h(x) m(dx)
\]

\[
= \int_0^\infty e^{-\alpha t} \int_E f(t; x, y) h(x) m(dx).
\]

By Laplace inversion,

\[
(5.5) \quad q^1_y(t; x) = f(t; x, y)
\]

for \(dt \otimes dm\)-a.e. \((t, x)\) in \([0, +\infty[ \times |y, B|\). Since both sides of (5.5) are exit laws (and so excessive functions in time-space), it follows that (5.5) holds identically for \(t > 0, y \in E^\circ\) and \(E \ni x > y\). See §3 of [5], and especially (3.17) therein.

(5.6) Proposition. For \(b \in E, x \in E, u \in ]0, t[,\) and \(y \in ]A, b \wedge x[,\)

\[
(5.7) \quad P^b(H_t \in dy, \rho_t \in du, X_t \in dx) = f(u; b, y) f(t - u; x, y) ds(y) du m(dx).
\]
The proposition now follows from (5.2) and (5.5).

\[ \text{Moreover,} \]
\[ \{ g(X_u) \phi(u) h(X_{t-u} \circ \theta_u) J(t - u, X_u, e^u) \} \]
\[ \text{and only if} \]
\[ \{ g(X_u) \phi(u) h(e_{t-u}) J(t - u, X_u, e^u) \} \]
\[ = \int_{A} \int_{\Omega} g(y) \phi(T_y(\omega)) n_y^t (h(X_{t-T_y(\omega)}) P^b(d\omega)) \ ds(y) \]
\[ = \int_{A} \int_{\omega \uparrow 0} g(y) \phi(u) n_y^t (h(X_{t-u}) f(u; b, y)) \ du \ ds(y). \]

The proposition now follows from (5.2) and (5.5). \( \square \)

Our local decomposition of \( X \) will be expressed in terms of certain “bridges” of \( X \). First, let \( \hat{K}^{y,\ell,x} \) denote the \( h \)-transform of \( P^t_y \) by means of the time-space harmonic function

\[ h_{\ell,x}(t, z) = q^y(\ell - t; z, x) \left[ \frac{r(y) - r(x)}{r(y) - r(z)} \right] 1_{[0,\ell]}(t), \]

where \( \ell > 0 \) and \( x > y \). Straightforward computations show that the absolute probabilities and transition probabilities under \( \hat{K}^{y,\ell,x} \) are given by

\[ \hat{K}^{y,\ell,x}(X_t \in dz) = \frac{q^y(\ell - t; z, x) f(t; y, z)}{f(\ell; y, x)} m(dz), \]

and

\[ \hat{K}^{y,\ell,x}(X_{t+v} \in dw | X_t = z) = \frac{q^y(v; z, w) q^y(\ell - t - v; w, x)}{q^y(\ell - t; z, x)} m(dw). \]

Moreover (cf. [15])

\[ \hat{K}^{y,\ell,x}(\zeta = \ell, X_\zeta = x) = 1, \]

\[ \int_{[y,B]} \hat{K}^{y,\ell,x} (F) P^t_y (X_\ell \in dx) = P^t_y (F \circ k_\ell). \]

Thus, \( \{ \hat{K}^{y,\ell,x} : x \in [y,B] \} \) is a regular version of the conditional probabilities \( F \mapsto P^t_y (F \circ k_\ell | X_\ell = x) \).

Now let \( K^{x,\ell,y} \) denote the image of \( \hat{K}^{y,\ell,x} \) under the time-reversal mapping, taking \( \omega \) to the path \( \gamma_\ell \omega \) defined by

\[ (\gamma_\ell \omega)(t) = \begin{cases} 
\omega(\ell - t), & 0 < t < \ell \\
\omega(\ell -), & t = 0 \\
\Delta, & t \geq \ell.
\end{cases} \]
Like $\hat{K}^{y,\ell,x}$, $K^{x,\ell,y}$ is the law of a non-homogeneous Markov diffusion; from (5.8) we see that

$$K^{x,\ell,y}(X_0 = x, \zeta = \ell, X_{\zeta^-} = y) = 1.$$  

Moreover, computation of finite dimensional distributions shows that the transition probabilities for $K^{x,\ell,y}$ are given by

$$(5.9) \quad K^{x,\ell,y}(X_t + v \in dw | X_t = z) = \frac{q_y(v; z, w) f(\ell - t - v; w, y)}{f(\ell - t; z, y)}.$$  

It follows that $\{K^{x,\ell,y} : \ell > 0\}$ is a regular version of the conditional probabilities

$$P^x_y(\cdot | \zeta = \ell).$$

**(5.10) Theorem.** Let $b \in E$. Then under $P^b$ the path fragments $(X_t : 0 \leq t < \rho_t)$ and $(X_{\rho_t + u} : 0 \leq u < t - \rho_t)$ are conditionally independent given $(H_t, \rho_t, X_t)$ on $\{X_t \in E\} = \{t < \zeta\}$. Moreover, given that $H_t = y$, $\rho_t = u$, and $X_t = x$ ($0 < u < t$, $y > x$), the above processes have conditional laws $K^{b,u,y}$ and $\hat{K}^{y,t-u,x}$ respectively.

The proof of (5.10) is similar to that of (4.8) and is left to the interested reader as an exercise.

**6. A result of W. Vervaat**

In this last section we use the decomposition of §5 to give a new proof of a result of Vervaat [17] which concerns a path transformation carrying Brownian bridge into Brownian excursion.

In this section we take the basic process $(X_t, P^x)$ to be standard Brownian motion on $\mathbb{R}$. Let $P_0$ denote the law of *Brownian bridge*; namely,

$$P_0(F) = P^0(X_1 = 0), \quad F \in \mathcal{F}_1.$$  

Under $P_0$ the coordinate process is centered Gaussian with continuous paths, $X_0 = 0$, and covariance $P_0(X_uX_t) = u(1 - t)$ for $0 \leq u \leq t \leq 1$.

Next, Let $P_+$ denote the law of scaled Brownian excursion. Under $P_+$ the coordinate process $(X_t : 0 \leq t \leq 1)$ is a non-homogeneous Markov diffusion with absolute probabilities

$$(6.1)(a) \quad P_+(X_t \in dx) = \frac{2x^2}{\sqrt{2\pi t^3(1-t)^3}} e^{-x^2/2(1-t)^3}$$  

where $t = u$.  

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and transition probabilities

\[(6.1)(b) \quad P_+(X_{t+v} \in dy | X_t = x) = p(v; y - x) \left( \frac{1 - t}{1 - t - v} \right)^{3/2} \frac{y \exp(-y^2/2(1 - t - v))}{x \exp(-x^2/2(1 - t))}, \]

where \(p(v; x) = (2\pi v)^{-1/2} e^{-x^2/2v}\) is the Gauss kernel, and \(0 < t < t + v < 1, 0 < x, y\).

Also, \(P_+(\zeta = 1) = P_+(X_t > 0, \forall t \in [0, 1]) = P_+(X_0 = X_1 = 0) = 1\).

Computation of finite dimensional distributions now shows that the following identities hold:

\[k_u(P_+ (\cdot | x_u = y)) = \hat{K}^{0,u,y},\]
\[\theta_u(P_+ (\cdot | X_u = y)) = K^{y,1-u,0},\]

where \(\hat{K}^{0,u,y}\) and \(K^{y,1-u,0}\) are as defined in the last section, the basic process being standard Brownian motion.

Now let \(\Omega_0 = \{ \omega \in \Omega : \omega(0) = \omega(1-) = 0, \zeta(\omega) = 1 \}\) and \(\overline{\Omega} = \Omega_0 \times [0, 1]\). Define a map \(\Phi: \overline{\Omega} \to \Omega_0\) by

\[\Phi(\omega, u)(t) = \Phi_u(\omega)(t) = \begin{cases} \omega(u + t) - \omega(u), & 0 \leq t < 1 - u, \\ \omega(u + t - 1) - \omega(u), & 1 - u \leq t < 1. \end{cases}\]

In the following we regard \(P_+\) and \(P_0\) as measures on \(\Omega_0\). Define \(\overline{P}\) on \(\overline{\Omega}\) by \(\overline{P} = P_+ \otimes \lambda\), where \(\lambda\) is Lebesgue measure on \([0, 1]\).

\[\text{(6.2) Proposition. The joint law of } (\Phi, V, X_U) \text{ under } \overline{P} \text{ is the same as the joint law of } (\omega, \rho_1, -H_1) \text{ under } P_0.\]

\[\text{Proof.}\]

For paths \(\omega\) and \(\omega'\), and \(t \in [0, 1]\) let \(\omega/t/\omega'\) denote the spliced path

\[\frac{\omega}{t/\omega'}(u) = \begin{cases} \omega(u), & 0 \leq u < t, \\ \omega'(u - t), & t \leq u < 1, \end{cases}\]

and let \(\tau_{y}\omega(t) = \omega(t) - y\). Let \(p_+(u, y) = P_+(X_u \in dy)/dy\). Note that if \(\omega(u) = \omega'(0)\), then

\[\Phi(\omega/u/\omega', u) = (\tau_{y}\omega/1 - u/\tau_{y}\omega),\]

where \(0 < u < 1\) and \(y = \omega(u)\). Thus,

\[\overline{P}(F \circ \Phi \psi(V, X_U)) = \int_0^1 P_+(F \circ \Phi_u \psi(1 - u, X_u)) \, du\]
\[= \int_0^1 \int_0^\infty \int_{\Omega} \int_{\Omega} F(\Phi_u(\omega/u/\omega')) \psi(1 - u, y) \hat{K}^{0,u,y}(d\omega) K^{y,1-u,0}(d\omega') p_+(u, y) \, dy \, du\]
\[= \int_0^1 \int_0^\infty \int_{\Omega} \int_{\Omega} F(\tau_{y}\omega/1 - u/\tau_{y}\omega) \psi(1 - u, y) \hat{K}^{0,u,y}(d\omega) K^{y,1-u,0}(d\omega') p_+(u, y) \, dy \, du\]
\[= \int_0^1 \int_0^\infty \int_{\Omega} \int_{\Omega} F(\omega'/1 - u/\omega) \psi(1 - u, y) K^{0,1-u,-y}(d\omega') K^{y,u,0}(d\omega) p_+(u, y) \, dy \, du\]
\[= P_0(F \cdot \psi(\rho_1, -H_1)).\]
Corollary. (Vervaat): Define a transformation $\Psi : \Omega_0 \to \Omega_0$ by

$$(\Psi \omega)(t) = \begin{cases} 
\omega(\rho_1(\omega) + t) - H_1(\omega), & 0 \leq t < 1 - \rho_1(\omega), \\
\omega(\rho_1(\omega) + t + 1) - H_1(\omega), & 1 - \rho_1(\omega) \leq t < 1.
\end{cases}$$

Then $\Psi(P_0) = P_+$. That is, the $P_0$-law of $(X_t : 0 \leq t < 1)$ is $P_+$.

Proof. It is easy to check that $\Psi \circ \Phi(\omega, u) = \omega$ for all $(\omega, u) \in \overline{\Omega}$. Using Proposition (6.2),

$$P_0(F \circ \Psi) = \mathcal{P}(F \circ \Psi \circ \Phi)$$
$$= \mathcal{P}(F \circ \pi_1)$$
$$= P_+(F),$$

where $\pi_1 : (\omega, u) \to \omega$. 

References


