Excursions Above the Minimum for Diffusions *

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1. Introduction

Let $X = (X_t)_{t\geq 0}$ be a regular diffusion process on an interval $E \subset \mathbb{R}$. Let $H_t := \min_{0\leq u\leq t} X_u$ denote the past minimum process of X and consider the excursions of X above its past minimum level: If $[a, b]$ is a maximal interval of constancy of $t \mapsto H_t$, then $(X_t : a \leq t \leq b)$ is the "excursion above the minimum" starting at time a and level $y = H_a$. These excursions, when indexed by the level at which they begin, can be regarded (collectively) as a point process. The independent increments property of the first-passage process of X implies that this point process is Poisonnian in nature, albeit non-homogeneous in intensity. Moreover, intuition tells us that the distribution of an excursion above the minimum $(X_t : a \leq t \leq b)$ should be governed by the Itô excursion law corresponding to excursions above the fixed level $y = H_a(\omega)$.

Our first task is to render precise the ruminations of the preceding paragraph. This is accomplished in sections 2 and 3 by applying Maisonneuve's theory of exit systems [**10**] to a suitable auxiliary process $(\overline{X}_t)$ associated with X. The basic result, stated in section 2, affirms the existence of a "Lévy system" for the point process of excursions of X above its past minimum.

In sections 4, 5, and 6 we discuss several applications of the Lévy system constructed in section 3; these applications concern path decompositions of X involving the minimum process H. Such decompositions, and related results, have been found by various authors (see [**6, 9, 11, 12, 14, 15, 16, 17, 18**]), most often in the special case where X is Brownian motion. The possibility of using Lévy systems to give a unified treatment of path decompositions is, of course, not surprising. In an excellent synthesis [**13**] Pitman has shown how the existence of a Lévy system for a point process attached to a Markov process leads naturally to various path decompositions of the Markov process.

In section 4 we obtain a general version of Williams' decomposition of a diffusion at its global minimum. A "local" version of Williams' decomposition can be found in section 5. In section 6 we give a new proof of a result of Vervaat [**17**], which states that a Brownian bridge, when split at its minimum and suitably " rearranged" becomes a (scaled) Brownian

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excursion. Indeed, we produce an inversion of Vervaat's transformation, showing how a Brownian excursion may be split and rearranged to yield Brownian bridge.

2. Notation and the basic result

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x)$ be a canonically defined regular diffusion on an interval $E \subset \mathbb{R}$. Here Ω denotes the space of paths $\omega\colon [0, +\infty[\to E \cup \{\Delta\}$ which are absorbed in the cemetery point $\Delta \notin E$ at time $\zeta(\omega)$, and which are continuous on $[0, \zeta(\omega)[$. For $t \geq 0$, $X_t(\omega) = \omega(t)$, and $\theta_t\omega$ denotes the path $u \mapsto \omega(u+t)$. The σ-fields $\mathcal{F}$ and $\mathcal{F}_t$ $(t \geq 0)$ are the usual Markovian completions of $\mathcal{F}^\circ = \sigma\{X_u : u \geq 0\}$ and $\mathcal{F}_t^\circ = \sigma\{X_u : 0 \leq u \leq t\}$ respectively. The law P^x on $(\Omega, \mathcal{F}^\circ)$ corresponds to X started at $x \in E$. We shall also make use of the killing operators (k_t) defined for $t \geq 0$ by

$$k_t\omega(u) = \begin{cases} \omega(u), & u < t, \\ \Delta, & u \geq t. \end{cases}$$

Let $A = \inf E$, $B = \sup E$, and write $E^\circ =]A, B[$. We assume throughout the paper that $A \notin E$, and that $B \in E$ if and only if B is a regular boundary point which is not a trap for X. In particular, these assumptions imply that the transition kernels of X are absolutely continuous with respect to the speed measure m (recalled below). See §4.11 of Itô-McKean [**8**].

Let s (resp. m, resp. k) denote a scale function (resp. speed measure, resp. killing measure) for X. Recall from [**8**] that the generator $\mathcal{G}$ of X has the form

$$\mathcal{G}u(x) \cdot m(dx) = du^+(x) - u(x) \cdot k(dx), \qquad x \in E^\circ, \tag{2.1}$$

for $u \in D(\mathcal{G})$, the domain of $\mathcal{G}$. Here and elsewhere u^+ denotes the scale derivative:

$$u^+(x) = \lim_{y \downarrow x} \frac{u(y) - u(x)}{s(y) - s(x)}.$$

Let $(U^\alpha : \alpha > 0)$ denote the resolvent family of X. Subsequent calculations require an explicit expression for the density of $U^\alpha(x, dy)$ with respect to $m(dy)$. Recall from [**8**] that for each $\alpha > 0$ there are strictly positive, linearly independent solutions g_1^α and g_2^α of

$$\mathcal{G}g(x) = \alpha g(x), \qquad x \in E^\circ; \tag{2.2}$$

g_1^α (resp. g_2^α) is an increasing (resp. decreasing) solution of (2.2) which also satisfies the appropriate boundary condition at A (resp. B). Both g_1^α and g_2^α are uniquely determined

up to a positive multiple. We sometimes drop the superscript α, writing simply g_1 and g_2. Since g_1 and g_2 are linearly independent solutions of (2.2), the Wronskian $W = g_1^+ g_2 - g_2^+ g_1$ is constant. The resolvent U^α is given by

$$U^\alpha f(x) = U^\alpha(x, f) = \int_E u^\alpha(x, y) f(y)\, m(dy), \tag{2.3}$$

where

$$u^\alpha(x, y) = u^\alpha(y, x) = g_1^\alpha(x) g_2^\alpha(y)/W, \qquad x \le y. \tag{2.4}$$

See §4.11 of [**8**] and note that in (2.3) the mass $m(\{B\})$ is the "stickiness" coefficient occurring in the boundary condition at B for elements of $D(\mathcal{G})$.

A jointly continuous version $(L_t^y : t \ge 0, y \in E)$ of *local time* for X may be chosen, and normalized to be occupation density relative to m, so that

$$P^x \int_0^\infty e^{-\alpha t}\, dL_t^y = u^\alpha(x, y). \tag{2.5}$$

Fixing a level $y \in E$, the local time $(L_t^y : t \ge 0)$ is related to the Itô excursion law [**7**], for excursions from level y, as follows. Let $G(y)$ denote the (random) set of left-hand endpoints (in $]0, \zeta[$) of intervals contiguous to the level set $\{t > 0 : X_t = y\}$. Define the hitting time T_y by

$$T_y = \inf\{t > 0 : X_t = y\} \qquad (\inf \emptyset = +\infty).$$

The Itô excursion law n_y is determined by the identity

$$P^x \sum_{u \in G(y)} Z_u\, F \circ k_{T_y} \circ \theta_u = P^x \left(\int_0^\infty Z_u\, dL_u^y \right) \cdot n_y(F), \tag{2.6}$$

where $x \in E$, $F \in p\mathcal{F}^\circ$, and $Z \ge 0$ is an $(\mathcal{F}_t)$-optional process. Under n_y the coordinate process $(X_t : t > 0)$ is strongly Markovian with semigroup (Q_t^y) given by

$$Q_t^y(x, f) = P^x(f \circ X_t; t < T_y). \tag{2.7}$$

The entrance law $n_y(X_t \in dz)$ is determined by the corresponding Laplace transform

$$W^\alpha f(y) = W^\alpha(y, f) = n_y \int_0^\zeta e^{-\alpha t} f \circ X_t\, dt. \tag{2.8}$$

Conversely, n_y is the unique σ-finite measure on $(\Omega, \mathcal{F}^\circ)$ which is carried by $\{\zeta > 0\}$ and under which $(X_t : t > 0)$ is Markovian with semigroup (2.7) and entrance law (2.8).

Let V_y^α denote the resolvent of the semigroup (Q_t^y). Taking $Z_u = e^{-\alpha u}$, $F = \int_0^\zeta e^{-\alpha t} f(X_t)\,dt$ in (2.6), and using (2.5), we obtain the important identity

$$U^\alpha f(x) = V_y^\alpha f(x) + u^\alpha(x,y)[m(\{y\})f(y) + W^\alpha f(y)]. \tag{2.9}$$

We also recall from §4.6 of [**8**] that the distribution of T_y is given by

$$P^x(e^{-\alpha T_y}) = \begin{cases} g_1^\alpha(x)/g_1^\alpha(y), & x \le y, \\ g_2^\alpha(x)/g_2^\alpha(y), & x \ge y. \end{cases} \tag{2.10}$$

Finally, the point process of excursions above the minimum is defined as follows. For $t \ge 0$ set

$$\begin{aligned} H_t(\omega) &= \begin{cases} \min_{0\le u\le t} X_u(\omega) & \text{if } t < \zeta(\omega), \\ -\infty & \text{if } t \ge \zeta(\omega); \end{cases} \\ M(\omega) &= \{u > 0 : X_u(\omega) = H_u(\omega)\}; \\ R_t(\omega) &= \inf\{u > 0 : u + t \in M(\omega)\}; \\ G(\omega) &= \{u > 0 : u < \zeta(w), R_{u-}(\omega) = 0 < R_u(\omega)\}. \end{aligned}$$

Thus G is the random set of left-hand endpoints of intervals contiguous to the random set M. For $u \in G$ we have the excursion $\mathbf{e}^u$ defined by

$$\mathbf{e}_t^u = \begin{cases} X_{u+t}, & 0 \le t < R_u, \\ \Delta, & t \ge R_u. \end{cases}$$

The point process $\Pi = (\mathbf{e}^u : u \in G)$ admits a Lévy system as follows. Define a continuous increasing adapted process $C = (C_t : t \ge 0)$ by

$$C_t = \begin{cases} s(H_0) - s(H_t), & \text{if } t < \zeta, \\ C_{\zeta-} & \text{if } t \ge \zeta. \end{cases}$$

(2.11) Theorem. *For $Z \ge 0$ and $(\mathcal{F}_t)$-optional, and $F \in p\mathcal{F}^\circ$,*

$$\begin{aligned} P^x \sum_{u\in G} Z_u\, F(\mathbf{e}^u) &= P^x \int_0^\infty Z_u\, n_{X_u}^\uparrow(F)\, dC_u \\ &= P^x \int_A^x Z_{T_y} 1_{\{T_y<+\infty\}} n_y^\uparrow(F)\, ds(y), \end{aligned} \tag{2.12}$$

where $n_y^\uparrow$ denotes the restriction of n_y to $\{\omega : \omega(t) > y, \forall t \in]0, \zeta(\omega)[\}$.

(2.13) Remark. The second equality in (2.12) follows from the first by the change of variable $u = T_y$. The equality of the first and third terms in (2.12) amounts to the

statement that the time-changed point process $(\mathbf{e}^{T_y} : R_{T_y-} < R_{T_y}, A < y < x)$ is a stopped Poisson point process under P^x, with (non-homogeneous) intensity $ds(y)n_y^{\uparrow}(d\omega)$, stopped at the first level y for which $T_y = +\infty$. See [**14**] for this result in the case of Brownian motion, with or without drift. The general result (2.11) was suggested by §4.10 of [**8**].

3. Proof of Theorem (2.11)

Maisonneuve's theory of exit systems [**10**] provides a Lévy system description of the point process of excursions induced by a closed, optional, homogeneous random set. Unfortunately the set M introduced in §2 is not (θ_t)-homogeneous; however the theory of [**10**] can be brought to bear once we note that M is homogeneous as a functional of the strong Markov process (X_t, H_t), $t \geq 0$. This key observation is due to Millar [**12**] and has been formalized by Getoor in [**4**]. In the terminology of [**4**], the process H is a "min-functional": $H_{t+u} = H_t \wedge H_u \circ \theta_t$. This property ensures that $\overline{X} := (X, H)$ is Markovian, as a simple computation shows.

Following [**4**] we first construct a convenient realization of $\overline{X}$. Let $\overline{\Omega} = \Omega \times (E \cup \{-\infty\})$, $\overline{E} = \{(x, a) \in E \times E : a \leq x\}$, and for $(\omega, a) \in \overline{\Omega}$ set

$$\overline{X}_t(\omega, a) = (X_t(\omega), a \wedge H_t(\omega)),$$
$$\overline{\theta}_t(\omega, a) = (\theta_t(\omega), a \wedge H_t(\omega)).$$

Clearly $\overline{X}_t \circ \overline{\theta}_u = \overline{X}_{t+u}$, $\overline{\theta}_t \circ \overline{\theta}_u = \overline{\theta}_{t+u}$. Moreover, M can be realized over $\overline{X}$ as

$$\overline{M}(\omega, a) = \{t > 0 : \overline{X}_t(\omega, a) \in D\}, \tag{3.1}$$

where $D = \{(x, x) : x \in E\}$. Let $\overline{\mathcal{F}}^\circ = \sigma\{\overline{X}_u : u \geq 0\}$, $\overline{\mathcal{F}}_t^\circ = \sigma\{\overline{X}_u : 0 \leq u \leq t\}$, and for $(x, a) \in \overline{E}$ let $\overline{P}^{x,a} = P^x \otimes \epsilon_a$. The usual Markovian completion of the filtration $(\overline{\mathcal{F}}_t^\circ)$ relative to the laws $(\overline{P}^{x,a} : (x, a) \in \overline{E})$ is denoted by $(\overline{\mathcal{F}}_t)$. Clearly $\overline{P}^{x,a}(\overline{X}_0 = (x, a)) = 1$ so that $\overline{X}$ has no branch points. Appealing to §2 of [**4**] we have the following

(3.2) Lemma. *(i) $\overline{X} = (\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{\theta}_t, \overline{X}_t, \overline{P}^{x,a})$ is a right-continuous, strong Markov process with state space $\overline{E}$ and cemetery $\overline{\Delta} = (\Delta, -\infty)$. The semigroup of $\overline{X}$ maps Borel functions to Borel functions, so that $\overline{X}$ is even a Borel right process.*

(ii) Let $\pi\colon (\omega, a) \to \omega$ denote the projection of $\overline{\Omega}$ onto Ω. If Z is an $(\mathcal{F}_t)$-optional process, then $Z \circ \pi$ is $(\overline{\mathcal{F}}_t)$-optional.

Now $\overline{M}$ is an $(\overline{\mathcal{F}}_t)$-optional, $(\overline{\theta}_t)$-homogeneous set, and each section $\overline{M}(\omega, a)$ is closed in $]0, \overline{\zeta}(\omega, a)[$. Set $\overline{R} = \inf \overline{M}$, so that $\overline{R}$ is an exact terminal time of $\overline{X}$ with $\mathrm{reg}(\overline{R}) =$

$\{(x,a) \in \overline{E} : \overline{P}^{x,a}(\overline{R}=0)=1\} = D$. This last fact follows from the regularity of X and the identity

$$\overline{P}^{x,a}(\overline{R} = T_a \circ \pi) = 1, \qquad (x,a) \in \overline{E}.$$

Let $\overline{G}$ denote the set of left-hand endpoints of intervals contiguous to $\overline{M}$. The properties of the Maisonneuve exit system $({}^*\overline{P}^{x,a}, \overline{K})$ for $\overline{M}$ are summarized in the next proposition. In what follows, $\overline{\mathcal{E}}^*$ and $\overline{\mathcal{F}}^*$ denote the universal completions of $\overline{\mathcal{E}}$ (the Borel sets in $\overline{E}$) and $\overline{\mathcal{F}}^\circ$ respectively.

(3.3) Proposition. [Maisonneuve] *There exists a continuous additive functional (CAF), $\overline{K}$, of $\overline{X}$ with a finite 1-potential, and a kernel ${}^*\overline{P}^{x,a}$ from $(\overline{E}, \overline{\mathcal{E}}^*)$ to $(\overline{\Omega}, \overline{\mathcal{F}}^*)$ such that*

$$\overline{P}^{x,a} \sum_{u \in \overline{G}} \overline{Z}_u \overline{F}_u \circ \overline{\theta}_u = \overline{P}^{x,a} \int_0^\infty \overline{Z}_u \, {}^*\overline{P}^{\overline{X}_u}(\overline{F}_u)\, d\overline{K}_u, \tag{3.4}$$

whenever $\overline{Z} \geq 0$ is $(\overline{\mathcal{F}}_t)$-optional and $(u, \overline{\omega}) \mapsto \overline{F}_u(\overline{\omega})$ is a $\mathcal{B}_{[0,+\infty[} \otimes \overline{\mathcal{F}}^*$-measurable, positive function. The CAF $\overline{K}$ is carried by D. For each $(x,a) \in \overline{E}$, ${}^*\overline{P}^{x,a}$ is a σ-finite measure on $(\overline{\Omega}, \overline{\mathcal{F}}^*)$ under which the coordinate process is strongly Markovian with the same transition semigroup as $\overline{X}$.

(3.5) Remarks. The version of $({}^*\overline{P}^{x,a}, \overline{K})$ cited in (3.3) is a variant of that constructed in [**10**]; the difference stems from the possibility that $\overline{P}^{x,a}(\overline{\zeta} < +\infty)$ may be positive. The fact that $\overline{K}$ is continuous (and so carried by $D = \mathrm{reg}(\overline{R})$) follows from the construction in [**10**], since $\overline{M} = \{t > 0 : \overline{X}_t \in D\}$ and D is finely perfect (with respect to $\overline{X}$). Renormalizing the kernel ${}^*\overline{P}^{x,a}$ if necessary, we can and do assume that ${}^*\overline{P}^{y,y}(1 - e^{-\overline{R}}) = 1$ for all $y \in E$.

Our plan is to prove Theorem (2.11) by identifying ${}^*\overline{P}^{x,a}$ and $\overline{K}$ explicitly, thereby deducing (2.12) from (3.4). First note that by taking $x = y$ in (2.9) and using (2.4) we have

$$W^\alpha f(y) = \int_{]A,y[} [g_1^\alpha(z)/g_1^\alpha(y)] f(z)\, m(dz) + \int_{]y,B]} [g_2^\alpha(z)/g_2^\alpha(y)] f(z)\, m(dz), \tag{3.6}$$

where $y \in E$, $\alpha > 0$, and $f \geq 0$ is Borel measurable on E.

To identify $\overline{K}$ we define a second CAF of $\overline{X}$, $\overline{C}$, by the formula

$$\overline{C}_t(\omega, a) = \begin{cases} s(a \wedge H_0(\omega)) - s(a \wedge H_t(\omega)) & \text{if } t < \overline{\zeta}(\omega, a), \\ \overline{C}_{\overline{\zeta}-}(\omega, a) & \text{if } t \geq \overline{\zeta}(\omega, a); \end{cases}$$

and notice that $\overline{C}_t(\omega, X_0(\omega)) = C_t(\omega)$. Clearly the fine support of $\overline{C}$ is D.

For $x \in E$ put $\psi(x) = W 1_{]x,B]}(x)$.

(3.7) Proposition. *The CAFs $\overline{K}$ and $\int_0^t \psi(X_u)\, d\overline{C}_u$ are equivalent.*

Proof. By [**1**; IV(2.13)] it suffices to check that the CAFs in question have the same finite 1-potential (over $\overline{X}$). An argument of Vervaat [**17**] shows that $P^x(t \in M) = 0$ for all $x \in E$ and all $t > 0$. Consequently, $\overline{P}^{x,a}(t \in \overline{M}) = 0$ for all $(x, a) \in \overline{E}$ and all $t > 0$. By Fubini's theorem, $\int_0^\infty e^{-t} 1_{\overline{M}}(t)\, dt = 0$ a.s. $\overline{P}^{x,a}$ for all $(x, a) \in \overline{E}$. Thus taking $\overline{Z}_u(\overline{\omega}) = e^{-u}$, $\overline{F}_u(\overline{\omega}) = 1 - \exp(-\overline{R}(\overline{\omega}) \wedge \overline{\zeta}(\overline{\omega}))$ in (3.4), we may compute

$$
\begin{aligned}
\overline{P}^{x,a} \int_0^\infty e^{-u}\, d\overline{K}_u &= \overline{P}^{x,a} \sum_{u \in \overline{G}} e^{-u} \left(\int_0^{\overline{R} \wedge \overline{\zeta}} e^{-t} \right) \circ \overline{\theta}_u \\
&= \overline{P}^{x,a} \int_{\overline{R} \wedge \overline{\zeta}}^{\overline{\zeta}} e^{-u}\, du \\
&= \overline{P}^{x,a} (e^{-\overline{R} \wedge \overline{\zeta}} - e^{-\overline{\zeta}}) \\
&= P^x (e^{-T_a \wedge \zeta} - e^{-\zeta}) \\
&= P^x \int_0^\zeta e^{-t}\, dt - P^x \int_0^{T_a \wedge \zeta} e^{-t}\, dt \\
&= U^1 1(x) - V_a^1 1(x) \\
&= \frac{u^1(x, a)}{u^1(a, a)} U^1 1(a),
\end{aligned}
\tag{3.8}
$$

where the last equality follows easily from (2.9). On the other hand, our hypothesis regarding the boundary A implies that $g_1^1(A+)/g_2^1(A+) = 0$ (see [**8**; §4.6]). Thus

$$
\begin{aligned}
\overline{P}^{x,a} \int_0^\infty e^{-t} \psi(X_t)\, d\overline{C}_t &= P^x \int_{T_a}^\infty e^{-t} \psi(X_t)\, dC_t \\
&= P^x \int_A^a e^{-T_y} \psi(y)\, ds(y) \\
&= \int_A^a [g_2^1(x)/g_2^1(y)] \psi(y)\, ds(y).
\end{aligned}
\tag{3.9}
$$

In (3.9) we have used the change of variables $t = T_y$ to obtain the second equality, and (2.1) to obtain the third. Now from the definition of the Wronskian W we see that $d(g_1/g_2) = W \cdot [g_2]^{-2} ds$. Using this fact and the expression for ψ provided by (3.6) we

may continue the computation begun in (3.9) with

$$
\begin{aligned}
&= \int_A^a [g_2(x)/g_2(y)] \int_{]y,B]} [g_2(z)/g_2(y)]\, m(dz)\, ds(y) \\
&= \int_A^a \int_{]y,B]} [g_2(x)g_2(z)/W]\, m(dz)\, d(g_1/g_2)(y) \\
&= \int_E \int_A^{z\wedge a} d(g_1/g_2)(y)[g_2(x)g_2(z)/W]\, m(dz) \\
&= \int_E [g_1(z\wedge a)/g_2(z\wedge a)] \cdot [g_2(x)g_2(z)/W]\, m(dz) \\
&= [u^1(x,a)/u^1(a,a)]U^1 1(a).
\end{aligned} \tag{3.10}
$$

The last equality in (3.10) follows from (2.3) and (2.4). In view of (3.8)–(3.10), we see that $\overline{K}$ and $\int_0^t \psi(X_s)\, c\overline{C}_s$ have the same finite 1-potential and so the proposition is proved. □

For $y \in E$ define a measure $\overline{Q}^y$ on $(\overline{\Omega}, \overline{\mathcal{F}}^\circ)$ by $\overline{Q}^y(F) = {}^*\overline{P}^{y,y}(F \circ \overline{k}_{\overline{R}})$, where $\overline{k}_t$ is the killing operator on $\overline{\Omega}$. Since ${}^*\overline{P}^{y,y}(\overline{R} \neq T_y \circ \pi) = 0$, the first coordinate of $\overline{X}$, namely $(X_t : t > 0)$, is Markovian under $\overline{Q}^y$, with (Q_t^y) as semigroup. Indeed, we claim that $\psi(y)\pi(\overline{Q}^y) = n_y^\uparrow$, at least for ds-a.e. $y \in E$. To verify this claim it suffices to compare the associated entrance laws.

(3.11) Lemma. *Let f be a bounded positive Borel function on E. Then for ds-a.e. $y \in E$ we have*

$$
\psi(y)\overline{Q}^y \int_0^{\overline{\zeta}} e^{-\alpha t} f(X_t)\, dt = W^\alpha f(y), \qquad \forall \alpha > 0. \tag{3.12}
$$

Proof. Fix f as in the statement of the lemma and also fix $\alpha > 0$. For $y \in E$ write

$$
\gamma(y) = \overline{Q}^y \int_0^{\overline{\zeta}} e^{-\alpha t} f(X_t)\, dt.
$$

As noted in the proof of (3.7), we have $\int_0^\infty 1_{\overline{M}}(t)\, dt = 0$, $\overline{P}^{x,a}$-a.s. for all $(x,a) \in \overline{E}$. Thus,

for $x \in E$,

$$
\begin{aligned}
U^\alpha f(x) &= \overline{P}^{x,x} \sum_{u \in \overline{G}} e^{-\alpha u} \left(\int_0^{\overline{R}} e^{-\alpha t} f(X_t)\, dt \right) \circ \overline{\theta}_u \\
&= \overline{P}^{x,x} \int_0^\infty e^{-\alpha u} *\overline{P}^{\overline{X}_u} \left(\int_0^{\overline{R}} e^{-\alpha t} f(X_t)\, dt \right) d\overline{K}_u \\
&= \overline{P}^{x,x} \int_0^\infty e^{-\alpha u} \gamma(X_u)\psi(X_u)\, d\overline{C}_u \\
&= P^x \int_A^x e^{-\alpha T_y} \gamma(y)\psi(y)\, ds(y) \\
&= \int_A^x [g_2^\alpha(x)/g_2^\alpha(y)]\gamma(y)\psi(y)\, ds(y).
\end{aligned} \tag{3.13}
$$

On the other hand, by (2.3) and (2.4), we have

$$
U^\alpha f(x) = \int_{]A,x]} [g_1^\alpha(x) g_2^\alpha(y)/W] f(y)\, m(dy) + \int_{]x,B]} [g_1^\alpha(y) g_2^\alpha(x)/W] f(y)\, m(dy). \tag{3.14}
$$

If we equate the last line displayed in (3.13) with the right side of (3.14), divide the resulting identity by $g_2^\alpha(x)$, and then differentiate in x, we obtain

$$
\left[\int_{]x,B]} [g_2^\alpha(y)/g_2^\alpha(x)] f(y)\, m(dy) \right] ds(x) = \gamma(x)\psi(x)\, ds(x)
$$

as measures on E, and the lemmas follows. □

(3.15) Corollary. *For ds-a.e. $y \in E$, $\psi(y)\pi(\overline{Q}^y) = n_y^\uparrow$ as measures on $(\Omega, \mathcal{F}^\circ)$.*

Proof. As noted earlier, both $\psi(y)\pi(\overline{Q}^y)$ and $n_y^\uparrow$ make the coordinate process on $(\Omega, \mathcal{F}^\circ)$ into a Markov process with transition semigroup (Q_t^y). It follows from Lemma (3.11) that these measures have the same one-dimensional distributions (and consequently the same finite dimensional distributions) for ds-a.e. y. Since $\mathcal{F}^\circ = \sigma(X_u : u \geq 0)$ is countably generated, the corollary follows. □

Proof of Theorem (2.11). Let $Z \geq 0$ be $(\mathcal{F}_t)$-optional and let $F \geq 0$ be $\mathcal{F}^\circ$-measurable. By (3.2)(ii), the process $Z \circ \pi$ is $(\overline{\mathcal{F}}_t)$-optional. We may now use (3.4), (3.7), and (3.15) to

compute

$$\begin{aligned}
P^x \sum_{u\in G} Z_u F(\mathbf{e}^u) &= \overline{P}^{x,x} \sum_{u\in \overline{G}} Z_u\circ\pi F(\pi\circ\overline{k}_{\overline{R}}\circ\overline{\theta}_u) \\
&= \overline{P}^{x,x} \int_0^\infty Z_u\circ\pi \; {}^*\overline{P}^{\overline{X}_u}(F(\pi\circ\overline{k}_{\overline{R}}))\, d\overline{K}_u \\
&= \overline{P}^{x,x} \int_0^\infty Z_u\circ\pi \; \overline{Q}^{\overline{X}_u}(F(\pi\circ\overline{k}_{\overline{R}}))\, \psi(X_u)\, d\overline{C}_u \\
&= P^x \int_0^\infty Z_u \, n^{\uparrow}_{X_u}(F)\, dC_u.
\end{aligned}$$

The proof of Theorem (2.11) is complete. □

4. Williams' decomposition

In this section we use the Lévy system (2.12) to obtain a new proof (of a general version) of Williams' decomposition [**18**] of a diffusion at its global minimum. A more "computational" proof of Williams' theorem, based on the same idea used in the present paper, may be found in [**3**].

For simplicity we assume that $\gamma := H_{\zeta-}$ satisfies $P^x(\gamma > A) = 1$ for all $x \in E$. We also assume that $\rho := \inf\{t > 0 : X_t = \gamma\}$ satisfies $P^x(\rho < \zeta) = 1$ for all $x \in E$. Then ρ is the unique time at which X takes its global minimum value γ (*cf.* [**17**]). Note that for $x \geq y$ (both in E),

$$P^x(\gamma > y) = P^x(T_y = +\infty). \tag{4.1}$$

Define a function r on E by

$$r(x) = \begin{cases} P^x(T_{x_0} < +\infty), & x \geq x_0, \\ [P^{x_0}(T_x < +\infty)]^{-1}, & x < x_0, \end{cases}$$

where $x_0 \in E^\circ$ is fixed but arbitrary. Clearly r is strictly positive and decreasing. Arguing as in [**8**; pp. 128–129] one may check that r is the unique positive decreasing solution of $\mathcal{G}r \equiv 0$ on E° which satisfies $r(x_0) = 1$ and the boundary condition at B. Note that

$$P^x(T_y < +\infty) = r(x)/r(y), \qquad x > y. \tag{4.2}$$

Before proceeding to the decomposition theorem we need a preliminary result.

(4.3) Lemma. *For $y \in E^\circ$ let $S_y = \inf\{t > 0 : X_{t-} = y\}$. Then*

$$n^{\uparrow}_y(S_y = +\infty) = -\frac{r^+(y)}{r(y)}, \qquad \forall y \in E^\circ.$$

(Recall that $r^+ = d^+r/ds^+$.)

Proof. Let q be an increasing solution of $\mathcal{G}q \equiv 0$ on E° such that q is linearly independent of r. We assume that q is normalized so that the Wronskian $q^+r - r^+q$ is identically 1. Fix $a < b$ both in E° and let v_{ab} denote the potential density (relative to m) of X killed at time $T_a \wedge T_b$. One checks that for $x \le y$,

$$v_{ab}(x,y) = v_{ab}(y,x) = \frac{D(a,x)D(y,b)}{D(a,b)},$$

where $D(x,y)$ is the determinant

$$\begin{vmatrix} q(y) & q(x) \\ r(y) & r(x) \end{vmatrix}.$$

Note that $D(x,y) > 0$ if $x < y$. Now let $y \in]a,b[$ and use (2.6) to compute

$$\begin{aligned} P^y(T_b < T_a) &= P^y \sum_{u \in G(y)} 1_{\{u < T_a \wedge T_b\}} 1_{\{T_b < T_y\}} \circ \theta_u \\ &= P^y(L^y_{T_a \wedge T_b})\, n_y(T_b < \zeta). \end{aligned}$$

But clearly $P^y(T_b < T_a) = D(a,y)/D(a,b)$ while $P^y(L^y_{T_a \wedge T_b}) = v_{ab}(y,y)$, so that

$$n_y(T_b < \zeta) = [D(a,y)/D(a,b)]/v_{ab}(y,y) = D(y,b)^{-1}.$$

Finally,

$$\begin{aligned} n_y^\uparrow(S_y = +\infty) &= \lim_{x \downarrow y} n_y^\uparrow(T_x < +\infty, S_y = +\infty) \\ &= \lim_{x \downarrow y} n_y^\uparrow(T_x < +\infty)\, P^x(T_y = +\infty) \\ &= \lim_{x \downarrow y} n_y(T_x < \zeta)[1 - r(x)/r(y)] \\ &= \lim_{x \downarrow y} \left[\frac{1}{r(y)} \cdot \frac{r(y) - r(x)}{s(x) - s(y)} \cdot \frac{s(x) - s(y)}{D(y,x)} \right] \\ &= -r(y)^{-1} \cdot r^+(y), \end{aligned}$$

since $\lim_{x \downarrow y}[s(x) - s(y)]/D(y,x)$ is the reciprocal of the Wronskian $q^+r - r^+q \equiv 1$. ∎

Now define probability laws on $(\Omega, \mathcal{F}^\circ)$ by

$$P_y^{x\downarrow}(F) = P^x(F \circ k_{T_y} | T_y < +\infty), \tag{4.4}$$

$$P_y^\uparrow(F) = n_y^\uparrow(F | S_y = +\infty), \tag{4.5}$$

whenever $x > y > A$. The coordinate process is a diffusion under any of these laws: $P_y^{x\downarrow}$ is the law of X started at x, conditioned to converge to A, and then killed at T_y; $P_y^{\uparrow}$ is the law of X started at y and conditioned to never return to y. These conditionings are accomplished by means of the appropriate h-transforms. In particular, the associated infinitesimal generators are given by

$$\mathcal{G}_y^{x\downarrow} f(z) = r(z)^{-1}\mathcal{G}(fr)(z), \qquad z > y; \tag{4.6}$$

$$\mathcal{G}_y^{\uparrow} f(z) = r_y(z)^{-1}\mathcal{G}(fr_y)(z), \qquad z > y, \tag{4.7}$$

where $r_y(z) = 1 - r(z)/r(y)$.

We can now state the general version of Williams' theorem. Recall that $\gamma = H_{\zeta-}$ and $\rho = \inf\{t > 0 : X_t = \gamma\}$.

(4.8) Theorem. *(a) The joint law of (γ, ρ, ζ) is given by*

$$P^x(f(\gamma)e^{-\alpha\rho-\beta\zeta}) = \int_A^x [g_2^{\alpha+\beta}(x)/g_2^{\alpha+\beta}(y)]f(y)P_y^{\uparrow}(e^{-\beta\zeta})\frac{-dr(y)}{r(y)}. \tag{4.9}$$

(b) For $F, G \in b\mathcal{F}^\circ$ and ψ bounded and Borel on E,

$$P^x(F\circ k_\rho\psi(\gamma)G\circ\theta_\rho) = P^x(P_\gamma^{x\downarrow}(F)\psi(\gamma)P_\gamma^{\uparrow}(G)). \tag{4.10}$$

(4.11) Remark. The intuitive content of (4.10) is that the processes $(X_t : 0 \le t < \rho)$ and $(X_{\rho+t} : 0 \le t < \zeta - \rho)$ are conditionally independent under P^x, given γ; and that the conditional distributions, given that $\gamma = y$, are $P_y^{x\downarrow}$ and $P_y^{\uparrow}$ respectively.

Proof of (4.8). Define $J(y, \omega) = 1_{\{S_y=+\infty\}}(\omega)$ and observe that $\rho(\omega) = u$ if and only if $u \in G(\omega)$ and $J(X_u(\omega), \mathbf{e}^u(\omega)) = 1$. Thus, using (2.11),

$$\begin{aligned} P^x(F\circ k_\rho\psi(\gamma)G\circ\theta_\rho) &= P^x \sum_{u\in G} F\circ k_u\psi(X_u)G(\mathbf{e}^u)J(X_u, \mathbf{e}^u) \\ &= \int_A^x P^x(F\circ T_y; T_y < +\infty)\psi(y)n_y^{\uparrow}(G\cdot J(y,\cdot))\,ds(y) \\ &= \int_A^x P^{x\downarrow}(F)\psi(y)P_y^{\uparrow}(G)P^x(T_y < +\infty)n_y^{\uparrow}(S_y = +\infty)\,ds(y). \end{aligned} \tag{4.12}$$

Taking $F = G = 1$ in (4.12) we see that

$$P^x(\gamma \in dy) = P^x(T_y < +\infty)n_y^{\uparrow}(S_y = +\infty)\,ds(y). \tag{4.13}$$

Now (4.13) substituted into the last line of (4.12) yields (4.10). To obtain (4.9) use (4.10) with $F = e^{-(\alpha+\beta)\zeta}$ and $G = e^{-\alpha\zeta}$, noting that $F \circ k_\rho = e^{-(\alpha+\beta)\rho}$ and $\zeta = \rho + \zeta \circ \theta_\rho$ (P^x-a.s.) since $\rho < \zeta$, P^x-a.s. Thus

$$\begin{aligned} P^x(f(\gamma)e^{-\alpha\rho}e^{-\beta\zeta}) &= P^x(f(\gamma)[e^{-(\alpha+\beta)\zeta}]\circ k_\rho\,[e^{-\beta\zeta}]\circ\theta_\rho) \\ &= P^x(P_\gamma^{x\downarrow}(e^{-(\alpha+\beta)\zeta})f(\gamma)P_\gamma^{\uparrow}(e^{-\beta\zeta})). \end{aligned}$$

Formula (4.9) now follows since

$$\begin{aligned} P_y^{x\downarrow}(e^{-(\alpha+\beta)\zeta}) &= P^x(e^{-(\alpha+\beta)T_y})/P^x(T_y < +\infty) \\ &= [g_2^{\alpha+\beta}(x)/g_2^{\alpha+\beta}(y)]/P^x(T_y < +\infty), \end{aligned}$$

and since $n_y^{\uparrow}(S_y = +\infty) = -r^+(y)/r(y)$ (Lemma (4.3)). □

(4.14) Corollary. $P^x(\rho \in dt, \gamma \in dy) = P^x(T_y \in dt)\dfrac{-dr(y)}{r(y)}$.

5. A local decomposition

Fix $t > 0$ and define

$$\rho_t = \inf\{u > 0 : X_u = H_t\} \wedge t.$$

Arguing as in [**17**] one can show that, almost surely on $\{t < \zeta\}$, ρ_t is the unique $u \in]0, t[$ such that $X_u = H_t$. Our purpose in this section is to describe the conditional distribution of $\{X_u : 0 \le u \le t\}$ under P^b, given that $H_t = y$, $\rho_t = u$, and $X_t = x$. This conditional distribution has been computed by Imhof [**6**] for the Brownian motion (and closely related processes). The joint law of (H_t, ρ_t, X_t), again in the case of Brownian motion, has been found by Shepp [**16**]. See also [**2, 9, 15**] for related results.

We begin by computing the joint law of (H_t, ρ_t, X_t). Recall from [**8**; §4.11] that the first passage distribution $P^x(T_y \in dv)$ has a density $f(v; x, y)$ on $]0, +\infty[$ relative to Lebesgue measure. Note that if we set $F_{t,y}(x) = P^x(t < T_y < +\infty)$, then (see [**8**; p. 154])

$$f(t; x, y) = -\frac{\partial}{\partial t}F_{t,y}(x) = \mathcal{G}F_{t,y}(x), \qquad x > y \in E, t > 0. \tag{5.1}$$

Applying $Q^y(z, dx)$ to both sides of (5.1) and integrating over $x \in]y, +\infty[\cap E$ (making use of the relation $Q_s^y\mathcal{G} = \mathcal{G}Q_s^y$ on $]y, +\infty[$), we obtain

$$f(t + s; z, y) = \int_{]y,+\infty[} Q_s^y(z, dx)f(t; x, y).$$

In other words, $(t,x) \mapsto f(t;x,y)$ is an exit law for the semigroup (Q_s^y).

Next, recall from [**8**; §4.11] that the semigroup (Q_t^y) has a density $q^y(t;x,z)$ (for $x \wedge z > y$) relative to the speed measure $m(dz)$; we have $q^y > 0$ on $]0,+\infty[\times(]y,B[)^2$ and $q^y(t;x,z) = q^y(t;z,x)$. The entrance law for $n_y^{\uparrow}$ can now be expressed as

$$n_y^{\uparrow}(X_t \in dx) = q_y^{\uparrow}(t;x)\, m(dx), \tag{5.2}$$

where

$$q_y^{\uparrow}(t;x) = \int_{]y,B]} n_y^{\uparrow}(X_{t-u} \in dz) q^y(u;z,x). \tag{5.3}$$

Substituting (5.2) into (5.3) and using the symmetry of q^y, we see that

$$q_y^{\uparrow}(t+u;x) = \int_{]y,B]} Q_u^y(x,dz) q_y^{\uparrow}(t;z). \tag{5.4}$$

But (5.4) means that $(t,x) \mapsto q_y^{\uparrow}(t;x)$ is also an exit law for (Q_t^y). Finally, using (3.6), if $\alpha > 0$ and h is positive, measurable, and vanishes off $]y,B]$, we may compute

$$\begin{aligned}
\int_0^\infty e^{-\alpha t} \int_E q_y^{\uparrow}(t;x) h(x)\, m(dx)\, dt &= W^\alpha h(y) \\
&= \int_{]y,B]} [g_2^\alpha(x)/g_2^\alpha(y)] h(x)\, m(dx) \\
&= \int_{]y,B]} P^x(e^{-\alpha T_y}) h(x)\, m(dx) \\
&= \int_0^\infty e^{-\alpha t} \int_E f(t;x,y) h(x)\, m(dx).
\end{aligned}$$

By Laplace inversion,

$$q_y^{\uparrow}(t;x) = f(t;x,y) \tag{5.5}$$

for $dt \otimes dm$-a.e. (t,x) in $]0,+\infty[\times]y,B]$. Since both sides of (5.5) are exit laws (and so excessive functions in time-space), it follows that (5.5) holds identically for $t > 0$, $y \in E^\circ$ and $E \ni x > y$. See §3 of [**5**], and especially (3.17) therein.

(5.6) Proposition. *For $b \in E$, $x \in E$, $u \in]0,t[$, and $y \in]A, b \wedge x[$,*

$$P^b(H_t \in dy, \rho_t \in du, X_t \in dx) = f(u;b,y) f(t-u;x,y)\, ds(y)\, du\, m(dx). \tag{5.7}$$

Proof. Let g, h, and ϕ be bounded positive Borel functions on $\mathbb{R}$, vanishing off E, E, and $]0,t[$, respectively. Put $J(v,y,\omega) = 1_{\{S_y > v\}}(\omega)$. Using (2.11) we have, since $u = \rho_t(\omega)$ if and only if $u \in G(\omega)$ and $J(t-u, X_u(\omega), \mathbf{e}^u(\omega)) = 1$,

$$\begin{aligned} P^b(g(H_t)\phi(\rho_t)h(X_t)) &= P^b \sum_{u\in G} g(X_u)\phi(u)h(X_{t-u}\circ\theta_u)J(t-u, X_u, \mathbf{e}^u) \\ &= P^b \sum_{u\in G} g(X_u)\phi(u)h(\mathbf{e}^u_{t-u})J(t-u, X_u, \mathbf{e}^u) \\ &= \int_A^b \int_\Omega g(y)\phi(T_y(\omega))n_y^\uparrow(h(X_{t-T_y(\omega)})P^b(d\omega)\, ds(y) \\ &= \int_A^b \int_{]0,t[} g(y)\phi(u)n_y^\uparrow(h(X_{t-u})f(u;b,y)\, du\, ds(y). \end{aligned}$$

The proposition now follows from (5.2) and (5.5). □

Our local decomposition of X will be expressed in terms of certain "bridges" of X. First, let $\hat{K}^{y,\ell,x}$ denote the h-transform of $P_y^\uparrow$ by means of the time-space harmonic function

$$h_{\ell,x}(t,z) = q^y(\ell - t; z, x)\left[\frac{r(y)-r(x)}{r(y)-r(z)}\right] 1_{]0,\ell[}(t),$$

where $\ell > 0$ and $x > y$. Straightforward computations show that the absolute probabilities and transition probabilities under $\hat{K}^{y,\ell,x}$ are given by

$$\hat{K}^{y,\ell,x}(X_t \in dz) = \frac{q^y(\ell - t; z, x)f(t; y, z)}{f(\ell; y, x)}\, m(dz),$$

and

$$\hat{K}^{y,\ell,x}(X_{t+v} \in dw | X_t = z) = \frac{q^y(v; z, w)q^y(\ell - t - v; w, x)}{q^y(\ell - t; z, x)}\, m(dw).$$

Moreover (*cf.* [**15**])

$$\hat{K}^{y,\ell,x}(\zeta = \ell, X_{\zeta-} = x) = 1, \tag{5.8}$$

$$\int_{]y,B]} \hat{K}^{y,\ell,x}(F)\, P_y^\uparrow(X_\ell \in dx) = P_y^\uparrow(F\circ k_\ell).$$

Thus, $\{\hat{K}^{y,\ell,x} : x \in]y,B]\}$ is a regular version of the conditional probabilities $F \mapsto P_y^\uparrow(F\circ k_\ell | X_\ell = x)$.

Now let $K^{x,\ell,y}$ denote the image of $\hat{K}^{y,\ell,x}$ under the time-reversal mapping, taking ω to the path $\gamma_\ell\omega$ defined by

$$(\gamma_\ell\omega)(t) = \begin{cases} \omega(\ell - t), & 0 < t < \ell \\ \omega(\ell-), & t = 0 \\ \Delta, & t \geq \ell. \end{cases}$$

Like $\hat{K}^{y,\ell,x}$, $K^{x,\ell,y}$ is the law of a non-homogeneous Markov diffusion; from (5.8) we see that

$$K^{x,\ell,y}(X_0 = x, \zeta = \ell, X_{\zeta-} = y) = 1.$$

Moreover, computation of finite dimensional distributions shows that the transition probabilities for $K^{x,\ell,y}$ are given by

$$K^{x,\ell,y}(X_{t+v} \in dw | X_t = z) = \frac{q^y(v; z, w) f(\ell - t - v; w, y)}{f(\ell - t; z, y)}. \tag{5.9}$$

It follows that $\{K^{x,\ell,y} : \ell > 0\}$ is a regular version of the conditional probabilities

$$P_y^{x\downarrow}(\cdot | \zeta = \ell).$$

(5.10) Theorem. *Let $b \in E$. Then under P^b the path fragments $(X_t : 0 \le t < \rho_t)$ and $(X_{\rho_t+u} : 0 \le u < t - \rho_t)$ are conditionally independent given (H_t, ρ_t, X_t) on $\{X_t \in E\} = \{t < \zeta\}$. Moreover, given that $H_t = y$, $\rho_t = u$, and $X_t = x$ $(0 < u < t,\ y > x)$, the above processes have conditional laws $K^{b,u,y}$ and $\hat{K}^{y,t-u,x}$ respectively.*

The proof of (5.10) is similar to that of (4.8) and is left to the interested reader as an exercise.

6. A result of W. Vervaat

In this last section we use the decomposition of §5 to give a new proof of a result of Vervaat [**17**] which concerns a path transformation carrying Brownian bridge into Brownian excursion.

In this section we take the basic process (X_t, P^x) to be standard Brownian motion on $\mathbb{R}$. Let P_0 denote the law of *Brownian bridge*; namely,

$$P_0(F) = P^0(F | X_1 = 0), \qquad F \in \mathcal{F}_1.$$

Under P_0 the coordinate process is centered Gaussian with continuous paths, $X_0 = 0$, and covariance $P_0(X_u X_t) = u(1-t)$ for $0 \le u \le t \le 1$.

Next, Let P_+ denote the law of scaled Brownian excursion. Under P_+ the coordinate process $(X_t : 0 \le t \le 1)$ is a non-homogeneous Markov diffusion with absolute probabilities

$$P_+(X_t \in dx) = \frac{2x^2}{\sqrt{2\pi t^3 (1-t)^3}} e^{-x^2/2t(1-t)} \tag{6.1)(a}$$

and transition probabilities

$$(6.1)(b) \quad P_+(X_{t+v} \in dy | X_t = x) = p(v; y-x) \left(\frac{1-t}{1-t-v} \right)^{3/2} \frac{y \exp(-y^2/2(1-t-v))}{x \exp(-x^2/2(1-t))},$$

where $p(v;x) = (2\pi v)^{-1/2} e^{-x^2/2v}$ is the Gauss kernel, and $0 < t < t+v < 1$, $0 < x, y$. Also, $P_+(\zeta = 1) = P_+(X_t > 0, \forall t \in]0,1[) = P_+(X_0 = X_{1-} = 0) = 1$.

Computation of finite dimensional distributions now shows that the following identities hold:

$$k_u(P_+(\cdot | x_u = y)) = \hat{K}^{0,u,y},$$
$$\theta_u(P_+(\cdot | X_u = y)) = K^{y,1-u,0},$$

where $\hat{K}^{0,u,y}$ and $K^{y,1-u,0}$ are as defined in the last section, the basic process being standard Brownian motion.

Now let $\Omega_0 = \{\omega \in \Omega : \omega(0) = \omega(1-) = 0, \zeta(\omega) = 1\}$ and $\overline{\Omega} = \Omega_0 \times]0,1[$. Define a map $\Phi: \overline{\Omega} \to \Omega_0$ by

$$\Phi(\omega, u)(t) = \Phi_u(\omega)(t) = \begin{cases} \omega(u+t) - \omega(u), & 0 \le t < 1-u, \\ \omega(u+t-1) - \omega(u), & 1-u \le t < 1. \end{cases}$$

In the following we regard P_+ and P_0 as measures on Ω_0. Define $\overline{P}$ on $\overline{\Omega}$ by $\overline{P} = P_+ \otimes \lambda$, where λ is Lebesgue measure on $]0,1[$. Set $U(\omega, u) = u$ and $V = 1 - U$ on $\overline{\Omega}$.

(6.2) Proposition. *The joint law of (Φ, V, X_U) under $\overline{P}$ is the same as the joint law of $(\omega, \rho_1, -H_1)$ under P_0.*

Proof. For paths ω and ω', and $t \in]0,1[$ let $\omega/t/\omega'$ denote the spliced path

$$(\omega/t/\omega')(u) = \begin{cases} \omega(u), & 0 \le u < t, \\ \omega'(u-t), & t \le u < 1, \end{cases}$$

and let $\tau_y \omega(t) = \omega(t) - y$. Let $p_+(u,y) = P_+(X_u \in dy)/dy$. Note that if $\omega(u) = \omega'(0)$, then

$$\Phi(\omega/u/\omega', u) = (\tau_y \omega'/1 - u/\tau_y \omega),$$

where $0 < u < 1$ and $y = \omega(u)$. Thus,

$$\begin{aligned}
\overline{P}(F \circ \Phi\, \psi(V, X_U)) &= \int_0^1 P_+(F \circ \Phi_u\, \psi(1-u, X_u))\, du \\
&= \int_0^1 \int_0^\infty \int_\Omega \int_\Omega F(\Phi_u(\omega/u/\omega'))\psi(1-u,y) \hat{K}^{0,u,y}(d\omega) K^{y,1-u,0}(d\omega') p_+(u,y)\, dy\, du \\
&= \int_0^1 \int_0^\infty \int_\Omega \int_\Omega F(\tau_y \omega'/1 - u/\tau_y \omega)\psi(1-u,y) \hat{K}^{0,u,y}(d\omega) K^{y,1-u,0}(d\omega') p_+(u,y)\, dy\, du \\
&= \int_0^1 \int_0^\infty \int_\Omega \int_\Omega F(\omega'/1 - u/\omega)\psi(1-u,y) K^{0,1-u,-y}(d\omega') K^{-y,u,0}(d\omega) p_+(u,y)\, dy\, du \\
&= P_0(F \cdot \psi(\rho_1, -H_1)).
\end{aligned}$$

□

(6.3) Corollary. *(Vervaat): Define a transformation* $\Psi : \Omega_0 \to \Omega_0$ *by*

$$(\Psi\omega)(t) = \begin{cases} \omega(\rho_1(\omega) + t) - H_1(\omega), & 0 \le t < 1 - \rho_1(\omega), \\ \omega(\rho_1(\omega) + t + 1) - H_1(\omega), & 1 - \rho_1(\omega) \le t < 1. \end{cases}$$

Then $\Psi(P_0) = P_+$. *That is, the* P_0*-law of* $(X_t \circ \Psi : 0 \le t < 1)$ *is* P_+.

Proof. It is easy to check that $\Psi \circ \Phi(\omega, u) = \omega$ for all $(\omega, u) \in \overline{\Omega}$. Using Proposition (6.2),

$$\begin{aligned} P_0(F \circ \Psi) &= \overline{P}(F \circ \Psi \circ \Phi) \\ &= \overline{P}(F \circ \pi_1) \\ &= P_+(F), \end{aligned}$$

where $\pi_1 : (\omega, u) \to \omega$. □

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