EXCURSION THEORY REVISITED

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May 10, 2005

In memory of J.L. Doob

Abstract. Excursions from a fixed point b are studied in the framework of a general Borel right process X, with a fixed excessive measure m serving as background measure; such a measure always exists if b is accessible from every point of the state space of X. In this context the left-continuous moderate Markov dual process $\widehat{X}$ arises naturally and plays an important role. This allows the basic quantities of excursion theory such as the Laplace-Lévy exponent of the inverse local time at b and the Laplace transform of the entrance law for the excursion process to be expressed as inner products involving simple hitting probabilities and expectations. In particular if X and $\widehat{X}$ are honest, then the resolvent of X may be expressed entirely in terms of quantities that depend only on X and $\widehat{X}$ killed when they first hit b.

1. Introduction

Let X be a nice Markov process and b an element of its state space. Let M be the closure of the random set $\{t : X_t = b\}$. Then the complement of M is the disjoint union of a countable number of open intervals, the excursion intervals from b. Excursion theory is concerned with the analysis of M and the behavior of X on the excursion intervals. This description of excursion theory comes from Chapter VI §42 of [RW87], to which we refer the reader for some history of the subject. In his seminal paper [I71], K. Itô introduced the Poisson process point of view for describing excursions from a point. Excellent presentations of this theory may be found in Chapter VI of [RW87], Chapter

1991 *Mathematics Subject Classification*. Primary 60J40, secondary 60J66.

Key words and phrases. Markov processes, excursion theory, right processes, moderate Markov processes.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

XII of [RY91], Chapter 3 of [B92], and the papers [R83], [R84] and [R89]. A closely related approach to excursions (from more general subsets of the state space) is based on the notion of an exit system, due to B. Maisonneuve [Ma75]; see also E.B. Dynkin [Dy71]. The exit system point of view was used in [G79] and we adopt it here.

Our interest in the subject was rekindled by a recent paper of Fukushima and Tanaka [FT04]. Working in the context of an m-symmetric diffusion they showed that some of the basic quantities of excursion theory could be expressed as inner products of simple hitting probabilities or expectations. These results were preliminary to the main subject of the Fukushima-Tanaka paper but were intriguing and of interest in their own right. The purpose of this paper is to show that analogous results hold for an arbitrary Borel right process X on a Lusin space E with a fixed excessive measure m serving as background measure. It is perhaps surprising that the mere existence of an excessive measure enables one to obtain results of considerable interest. In concrete examples there is usually a natural choice for m. Of course, if one does not assume symmetry then a dual process enters the picture. The dual that appears most naturally is the left-continuous moderate Markov dual process $\widehat{X}$ associated with m and X. Such a dual process always exists. The simplest way to understand $\widehat{X}$ is in terms of the corresponding Kuznetsov process. The relevant facts may be found in [Fi87], [DMM92] or [Ma93]; section 2 of [G99] contains a good summary. See also [FG03]. The reader who is not familiar with this theory may assume throughout that X and $\widehat{X}$ are *standard* processes in weak duality with respect to m, as in [GS84].

In section 2 we describe the hypotheses that are in force throughout the paper, and we recall the basic facts of the exit system approach to excursions. This section also contains the fundamental decomposition of the resolvent in terms of excursion-related ingredients. In Section 3 the key formula expressing the Laplace-Lévy exponent, $g(\lambda)$, of the inverse local time at a regular point b as an inner product is developed. It appears in Theorem (3.6) and states that $g(\lambda) = \delta + \lambda c(m)^{-1} \int \varphi^\lambda \widehat{\varphi}\, dm$, where $\varphi^\lambda = \mathbf{E}^\bullet[e^{-\lambda T_B}]$, $\widehat{\varphi} = \widehat{\mathbf{P}}^\bullet[\widehat{T}_b < \infty]$, and $\delta = \lim_{\lambda\to 0} g(\lambda)$. Here T_b (resp. $\widehat{T}_b$) is the hitting time of b by X (resp. $\widehat{X}$). This formula depends on a specific normalization of the local time at b, $\ell = (\ell_t)$, inasmuch as the Revuz measure of ℓ with respect to m is $c(m)\varepsilon_b$, ε_b being the unit mass at b. Additional expressions for the constant δ are contained in Theorem (3.15). Moreover, the Laplace transform of the entrance law governing the excursion process is shown to be given by an inner product $(f, \widehat{\varphi}^\lambda) = \int f \widehat{\varphi}^\lambda\, dm$. One consequence of all this is that $g(\lambda)$ and the resolvent (U^λ) of X may be expressed entirely in terms of quantities that depend only on the processes X and $\widehat{X}$ killed at T_b and $\widehat{T}_b$ respectively, at least if X and

$\widehat{X}$ are honest. In section 4 we extend an old result of Harris [H56], Silverstein [Si80], and Getoor [G79], showing that under a mild condition, the mean occupation measure of excursions gives rise to an excessive measure. This is the case, in particular, if b is accessible from every other point of the state space. In Section 5 we present several examples that illustrate the results in Section 3 and 4. In particular, the extent to which δ and the "stickiness" $\gamma = \lim_{\lambda\to\infty} g(\lambda)/\lambda$ may be varied while the Lévy measure of g and the entrance law of the excursion process remain fixed. Finally, in an appendix we present several facts about the left-continuous moderate Markov dual process $\widehat{X}$ that are needed in the body of the paper but are not readily available in the literature. Some of these results are of independent interest.

We close this introduction with a few words on notation. We shall use $\mathcal{B}_{]0,\infty[}$ to denote the Borel subsets of the half-line $]0,\infty[$. If $(F,\mathcal{F},\mu)$ is a measure space, then $b\mathcal{F}$ (resp. $p\mathcal{F}$) denotes the class of bounded real-valued (resp. $[0,\infty]$-valued) $\mathcal{F}$-measurable functions on F. For $f\in p\mathcal{F}$ we use $\mu(f)$ or $\langle\mu,f\rangle$ to denote the integral $\int_F f\,d\mu$; similarly, if $D\in\mathcal{F}$ then $\mu(f;D)$ denotes $\int_D f\,d\mu$. On the other hand $f\mu$ denotes the measure $f(x)\mu(dx)$ and $\mu|_D$ the restriction of μ to D. We write $\mathcal{F}^*$ for the universal completion of $\mathcal{F}$; that is $\mathcal{F}^* = \cap_\nu \mathcal{F}^\nu$, where $\mathcal{F}^\nu$ is the ν-completion of $\mathcal{F}$ and the intersection runs over all finite measures on $(F,\mathcal{F})$. If $(E,\mathcal{E})$ is a second measurable space and $K = K(x,dy)$ is a kernel from $(F,\mathcal{F})$ to $(E,\mathcal{E})$ (i.e., $F\ni x\mapsto K(x,A)$ is $\mathcal{F}$-measurable for each $A\in\mathcal{E}$ and $K(x,\cdot)$ is a measure on $(E,\mathcal{E})$ for each $x\in F$), then we write μK for the measure $A\mapsto\int_F \mu(dx)K(x,A)$ and Kf for the function $x\mapsto\int_E K(x,dy)f(y)$. Finally if f is a measurable function from $(F,\mathcal{F})$ to $(E,\mathcal{E})$, then $\nu = f(\mu)$, the image of μ under f, is the measure on $(E,\mathcal{E})$ defined by $\nu(B) = \mu[f^{-1}(B)]$ for $B\in\mathcal{E}$.

2. The Lévy Exponent of the Inverse Local Time

Throughout this paper $(P_t, t\ge 0)$ will denote a Borel right semigroup on a Lusin state space $(E,\mathcal{E})$, and $X = (X_t,\mathbf{P}^x)$ will denote a right continuous strong Markov process realizing (P_t). In general we shall use the standard notation for Markov processes without special mention; see, for example, [BG68], [DM87], [Sh88] and [G90]. In particular $U^\lambda = \int_0^\infty e^{-\lambda t}P_t\,dt$ denotes the resolvent of (P_t). We adopt the usual convention that a real-valued function f defined on E is extended to the cemetery point Δ by $f(\Delta) := 0$. For example, $P_t f(x) = \mathbf{P}^x[f(X_t)] = \mathbf{P}^x[f(X_t); t<\zeta]$ and $U^\lambda f(x) = \mathbf{P}^x\int_0^\zeta f(X_t)\,dt$, where ζ denotes the lifetime of X.

We suppose that $b\in E$ is a regular point; that is $\mathbf{P}^b[T_b = 0] = 1$, where $T_b := \inf\{t>0 : X_t = b\}$ is the hitting time of $\{b\}$. Define $\varphi^\lambda(x) = \mathbf{P}^x[e^{-\lambda T_b}]$ for $\lambda>0$, and set $\varphi = \lim_{\lambda\to 0}\varphi^\lambda = \mathbf{P}^\bullet[T_b<\infty]$. (To avoid trivialities, we assume throughout that φ does not vanish identically on $E \setminus \{b\}$.) Let $X^b = (X, T_b)$ denote X killed when it hits b, and let Q_t and V^λ denote the semigroup and resolvent of (X, T_b):

$$\begin{aligned} Q_t f &= \mathbf{P}^x[f(X_t); t < T_b], \\ V^\lambda f(x) &= \mathbf{P}^x \int_0^{T_b} e^{-\lambda t} f(X_t)\, dt = \int_0^\infty e^{-\lambda t} Q_f f(x)\, dt. \end{aligned} \tag{2.1}$$

The lifetime of (X, T_b) is $R = T_b \wedge \zeta$, and one may replace T_b in (2.1) by R without changing the integrals, because of our convention that functions vanish at Δ. Also, $T_b < \zeta$ if and only if $T_b < \infty$ so $R = T_b$ on $\{T_b < \infty\}$. This statement holds $\mathbf{P}^x$-a.s. for all $x \in E$, and as is customary we shall omit the qualifier a.s. where it is clearly required. A direct application of the strong Markov property at time T_b shows that

$$U^\lambda f = V^\lambda f + \varphi^\lambda \cdot U^\lambda f(b), \qquad \lambda > 0, f \in p\mathcal{E}^*. \tag{2.2}$$

Because b is a regular point, the singleton $\{b\}$ is not semipolar and consequently there exists a local time $\ell = \ell^b$ for X at b. This is a positive continuous additive functional (PCAF) of X, increasing only on the visiting set $\{t \ge 0 : X_t = b\}$. As such ℓ is uniquely determined up to a multiplicative constant. Define, for $\lambda > 0$ and $f \in p\mathcal{E}^*$,

$$U_\ell^\lambda f = \mathbf{P}^\bullet \int_0^\infty e^{-\lambda t} f(X_t)\, d\ell_t. \tag{2.3}$$

Especially important is the λ-potential of ℓ,

$$u^\lambda(x) = U_\ell^\lambda 1(x) = \mathbf{P}^x \int_0^\infty e^{-\lambda t}\, d\ell_t. \tag{2.4}$$

Note that this integral is really over $]0, \zeta[$ since the measure $d\ell_t$ is carried by this interval. Using the strong Markov property of X

$$U_\ell^\lambda f = \mathbf{P}^\bullet \int_{T_b}^\infty e^{-\lambda t} f(X_t)\, d\ell_t = \varphi^\lambda \cdot U_\ell^\lambda f(b), \tag{2.5}$$

and, taking $f \equiv 1, u^\lambda(x) = \varphi^\lambda(x) u^\lambda(b)$.

The inverse local time $\tau = (\tau(t))_{t \ge 0}$ is the right continuous inverse of ℓ:

$$\tau(t) := \inf\{s > 0 : \ell_s > t\}, \qquad t \ge 0.$$

It is standard that $(\tau(t))$ under the law $\mathbf{P}^b$ is a strictly increasing subordinator and that

$$\mathbf{P}^b[e^{-\lambda\tau(t)}] = e^{-t/u^\lambda(b)}, \qquad \lambda > 0, t \geq 0. \tag{2.6}$$

Thus $g(\lambda) := [u^\lambda(b)]^{-1}$ is the subordinator exponent of $(\tau, \mathbf{P}^b)$; as is well known $g(\lambda)$ takes the form

$$g(\lambda) = \delta + \gamma\lambda + \int_{]0,\infty[} (1 - e^{-\lambda t})\, \nu(dt) \tag{2.7}$$

where δ and γ are nonnegative constants and ν is a measure on $]0, \infty[$ with

$$\int_{]0,\infty[} (t \wedge 1)\, \nu(dt) < \infty.$$

The measure ν is the *Lévy measure* of $(\tau, \mathbf{P}^b)$. Observe that

$$\delta = \lim_{\lambda\to 0} g(\lambda), \qquad \gamma = \lim_{\lambda\to\infty} g(\lambda)/\lambda. \tag{2.8}$$

It was shown in [GS73] that

$$\int_0^t 1_b(X_s)ds = \gamma \cdot \ell_t, \qquad \forall t \geq 0, \tag{2.9}$$

$\mathbf{P}^x$-a.s. for all $x \in E$, and so $U^\lambda 1_b(b) = \gamma u^\lambda(b)$. (Here 1_b is the indicator of $\{b\}$.) As both ℓ_t and $\int_0^t 1_b(X_s)ds$ vanish on $[0, T_b]$, we even have $U^\lambda 1_b = \gamma u^\lambda$.

(2.10) Proposition. *Under $\mathbf{P}^b, \ell_\infty$ has an exponential distribution with parameter δ. In particular, $\delta = 0$ if and only if $\mathbf{P}^b[\ell_\infty = \infty] = 1$ if and only if $\mathbf{P}^b[\ell_\infty] = \infty$.*

Proof. Observe that $\{\ell_\infty > t\} = \{\tau(t) < \infty\}$. Therefore

$$\mathbf{P}^b[\ell_\infty > t] = \mathbf{P}^b[\tau(t) < \infty] = \lim_{\lambda\to 0} \mathbf{P}^b[e^{-\lambda\tau(t)}] = \lim_{\lambda\to 0} e^{-tg(\lambda)} = e^{-\delta t}, \quad t > 0,$$

by (2.6) and (2.8). $\square$

An efficient way to compute probabilities to do with the excursions of X from the regular point b is the associated (predictable) *exit system* $(\mathbf{P}^*, l)$; see [Ma75, §9]. Let M denote the closure in $[0, \infty[$ of the visiting set $\{t \geq 0 : X_t = b\}$, and let G be the set of strictly positive left endpoints of the maximal complementary intervals of M. Then there exists a σ-finite measure $\mathbf{P}^*$ on $(\Omega, \mathcal{F}^*)$, where $\mathcal{F}^*$ is the universal completion of $\mathcal{F}^0$, such that

$$\mathbf{P}^x\left[\sum_{s\in G} Z_s \cdot F\circ\theta_s\right] = \mathbf{P}^x\left[\int_0^\infty Z_s\, dl_s\right]\cdot \mathbf{P}^*[F], \qquad x\in E, \tag{2.11}$$

provided $Z\geq 0$ is a predictable process and $F\in p\mathcal{F}^*$. (As with the local time ℓ, the measure $\mathbf{P}^*$ is only determined up to a constant multiple: If $(\mathbf{P}^*,\ell)$ is an exit system then so is $(c^{-1}\mathbf{P}^*, c\cdot\ell)$ for any constant $c>0$.) Taking $Z_s = e^{-s}$, $F = 1-e^{-T_b}$ and then $F = 1_{\{T_b=0\}}$ in (2.11) we see that

$$\mathbf{P}^*[1-e^{-T_b}] < \infty \quad \text{and} \quad \mathbf{P}^*[T_b = 0] = 0. \tag{2.12}$$

Under $\mathbf{P}^*$ the process $(X_t)_{t>0}$ is strong Markov with semigroup (P_t). This is a (very) special case of a result of Maisonneuve [Ma75]. Define

$$Q_t^*(f) = \mathbf{P}^*[f(X_t); t<T_b] = \mathbf{P}^*[f(X_t); t<R], \qquad t>0, f\in p\mathcal{E}^*. \tag{2.13}$$

It is well known and easily verified that (Q_t^*) is an *entrance law* for (Q_t) (the semigroup of the killed process (X,T_b)); that is $Q_{t+s}^* = Q_t^*Q_s$. Also, for each $t>0$, the measure Q_t^* is carried by $E_b := E\setminus\{b\}$ and $Q_t^*1_{E_b} = \mathbf{P}^*[t<R]<\infty$ because (by (2.12))

$$\infty > \mathbf{P}^*[1-e^{-T_b}] \geq \mathbf{P}^*[1-e^{-R}] \geq (1-e^{-t})\mathbf{P}^*[t<R].$$

(2.14) Proposition. *Define*

$$V_*^\lambda(f) := \int_0^\infty e^{-\lambda t} Q_t^*(f)\, dt = \mathbf{P}^*\int_0^R e^{-\lambda t} f(X_t)\, dt, \quad \lambda\geq 0, f\in p\mathcal{E}^*. \tag{2.15}$$

Then

$$U^\lambda f(b) = \frac{\gamma f(b) + V_*^\lambda(f)}{g(\lambda)}, \qquad \lambda\geq 0, f\in p\mathcal{E}^*. \tag{2.16}$$

Proof. Let f and λ be as above. Then

$$U^\lambda f(b) = \mathbf{P}^b\int_0^\infty e^{-\lambda t} f(X_t)1_b(X_t)\, dt + \mathbf{P}^b\sum_{s\in G}\int_s^{s+R\circ\theta_s} e^{-\lambda t} f(X_t)\, dt.$$

The first term on the right is just $f(b)U^\lambda 1_b(b) = \gamma f(b)u^\lambda(b)$ because of (2.9). Using (2.11) the second term on the right reduces to

$$\mathbf{P}^b\sum_{s\in G} e^{-\lambda s}\left[\int_0^R e^{-\lambda t} f(X_t)\, dt\right]\circ\theta_s = V_*^\lambda(f)u^\lambda(b).$$

The assertion now follows because $u^\lambda(b) = [g(\lambda)]^{-1}$. □

The next result concerns the last exit time from b, namely $L := \sup M$ (with the convention that $\sup\emptyset = 0$). Observe that $\mathbf{P}^b[L>0] = \mathbf{P}^b[T_b<\infty] = 1$.

(2.17) Proposition. *If $\lambda > 0$, then*

$$\mathbf{P}^b[e^{-\lambda L}] = \mathbf{P}^*[T_b = \infty]/g(\lambda) \tag{2.18}$$

and

$$\mathbf{P}^b[e^{-\lambda L}; L = \zeta] = \mathbf{P}^*[\zeta = 0]/g(\lambda). \tag{2.19}$$

In particular,

$$\delta = \mathbf{P}^*[T_b = \infty] \geq \mathbf{P}^*[\zeta = 0], \tag{2.20}$$

and $\mathbf{P}^b[L < \infty]$ is 0 or 1 according as $\delta = 0$ or $\delta > 0$.

Proof. Observe that L is the unique point $s \in G$ for which $T_b \circ \theta_s = \infty$. Therefore, by (2.11),

$$\begin{aligned}\mathbf{P}^b[e^{-\lambda L}] &= \mathbf{P}^b \sum_{s \in G} e^{-\lambda s} 1_{\{T_b = \infty\}} \circ \theta_s \\ &= u^\lambda(b) \cdot \mathbf{P}^*[T_b = \infty],\end{aligned}$$

proving (2.18). A similar computation proves (2.19) once we notice that $s = L = \zeta < \infty$ if and only if $s \in G$ and $\zeta \circ \theta_s = 0$. Multiplying both sides of (2.18) by $g(\lambda)$ and then sending $\lambda \to 0$ we obtain (on account of (2.8)) $\delta \cdot \mathbf{P}^b[L < \infty] = \mathbf{P}^*[T_b = \infty]$, proving (2.20) when $\delta = 0$. As in the proof of (2.10), $\mathbf{P}^b[\tau(t) = \infty] = 1 - e^{-\delta t}$ for $t > 0$. Because the closed range of the subordinator τ coincides with the closure of the visiting set $\{t : X_t = b\}$, we have the inclusion $\{\tau(t) = \infty\} \subset \{L < \infty\}$, for each $t > 0$. Therefore, if $\delta > 0$,

$$\begin{aligned}\mathbf{P}^b[L < \infty] &\geq \mathbf{P}^b\left[\cup_{n=1}^\infty \{\tau(n) = \infty\}\right] \\ &= \uparrow \lim_n \mathbf{P}^b[\tau(n) = \infty] = \lim_n (1 - e^{-\delta n}) = 1,\end{aligned}$$

proving (2.20) in this case as well. The final assertion of the proposition follows from (2.18) and (2.20). $\square$

(2.21) Remarks. (a) Letting $\lambda \to 0$ in (2.19) we see that $\mathbf{P}^*[\zeta = 0] = \delta \cdot \mathbf{P}^b[\zeta = L < \infty]$. This implies that when $\delta > 0$ there is no killing at b (*i.e.*, $\mathbf{P}^b[L = \zeta < \infty] = 0$) if and only if $\mathbf{P}^*[\zeta = 0] = 0$.

(b) Because $g(\lambda)^{-1} = \mathbf{P}^b \int_0^\infty e^{-\lambda t}\, d\ell_t$, we can invert the Laplace transform in (2.18) to obtain

$$\mathbf{P}^b[L \leq t] = \delta \cdot \mathbf{P}^b[\ell_t], \qquad t \geq 0.$$

(2.22) Corollary. $\mathbf{P}^*[T_b \in A] = \nu(A)$, *for all* $A \in \mathcal{B}_{]0,\infty[}$.

Proof. Let $T = T_b$ during this proof, and recall that $\varphi = \mathbf{P}^\bullet[T < \infty]$. Evidently $R \le T$ and $R = T$ on $T < \infty$; together with the terminal time property of T this yields

$$\begin{aligned} V_*^\lambda(\varphi) &= \mathbf{P}^* \int_0^R e^{-\lambda t} 1_{\{T\circ\theta_t < \infty\}}\, dt \\ &= \mathbf{P}^* \int_0^T e^{-\lambda t} 1_{\{T<\infty\}}\, dt = \lambda^{-1}\mathbf{P}^*\left[1 - e^{-\lambda T}; T < \infty\right]. \end{aligned} \tag{2.23}$$

Thus, by (2.16),

$$g(\lambda)\cdot \lambda U^\lambda \varphi(b) = \gamma\lambda + \mathbf{P}^*[1 - e^{-\lambda T}; T < \infty], \qquad \lambda > 0. \tag{2.24}$$

On the other hand

$$\begin{aligned} \lambda U^\lambda \varphi(b) &= \lambda \mathbf{P}^b \int_0^\zeta e^{-\lambda t} 1_{\{T<\infty\}}\circ\theta_t\, dt \\ &= \lambda \mathbf{P}^b \int_0^\zeta e^{-\lambda t} 1_{\{L>t\}}\, dt \\ &= \mathbf{P}^b[1 - e^{-\lambda L}]. \end{aligned}$$

Combining this with (2.18), (2.20), and (2.24) we find that

$$g(\lambda) = \delta + \gamma\lambda + \mathbf{P}^*[1 - e^{-\lambda T}; T < \infty], \qquad \lambda > 0. \tag{2.25}$$

Now define $h(t) = \nu(]t,\infty[)$. Then

$$\lambda \int_0^\infty e^{-\lambda t} h(t)\, dt = \int_{]0,\infty[} (1 - e^{-\lambda t})\, \nu(dt)$$

by Fubini's theorem. On the other hand, by (2.25) and (2.7),

$$\begin{aligned} \lambda \int_0^\infty e^{-\lambda t} \mathbf{P}^*[t < T < \infty]\, dt &= \mathbf{P}^*\left[\int_0^T \lambda e^{-\lambda t}\, dt\, ; T < \infty\right] \\ &= \mathbf{P}^*\left[1 - e^{-\lambda T}; T < \infty\right] \\ &= \int_0^\infty (1 - e^{-\lambda t})\, \nu(dt). \end{aligned}$$

Both $h(t)$ and $\mathbf{P}^*[t < T < \infty]$ are finite and right continuous on $]0,\infty[$, hence equal functions of t by the uniqueness theorem for Laplace transforms. The assertion now follows from the monotone class theorem. $\square$

(2.26) Corollary. $\mathbf{P}^*[R=\infty] = \delta \cdot \mathbf{P}^b[\zeta=\infty]$.

Proof. Since $\lambda V_*^\lambda(1_{E_b}) = \mathbf{P}^*[1-e^{-\lambda R}]$, $\mathbf{P}^*[R=\infty] = \lim_{\lambda\to 0} \lambda V_*^\lambda(1_{E_b})$. Also, from (2.16), $\lambda U^\lambda 1(b) = [\gamma\lambda + \lambda V_*^\lambda(1_{E_b})]/g(\lambda)$. As $\lambda \to 0, \lambda U^\lambda 1(b) = \mathbf{P}^b[1-e^{-\lambda\zeta}] \to \mathbf{P}^b[\zeta=\infty]$ and $g(\lambda)\to\delta$. Hence $\mathbf{P}^*[R=\infty] = \delta\cdot\mathbf{P}^b[\zeta=\infty]$. □

3. Excursions and Duality

We now fix an excessive measure m on $(E,\mathcal{E})$ to serve as background measure. Thus m is σ-finite and $mP_t \le m$ for all t. It is known that $mP_t \uparrow m$ (setwise) as $t\downarrow 0$; see [DM87, XII 36-37]. (The existence of m is a mild assumption and, as we shall see in Theorem (4.5), not really an assumption at all if b is accessible from all points of E.)

Associated with m and (P_t) is the Kuznetsov process $((Y_t)_{t\in\mathbf{R}}, Q_m)$. We shall make no explicit use of this process except in the appendix, but we suppose (without loss of generality) that $(X_t,\mathbf{P}^x)$ is the realization of (P_t) described on p. 53 of [G90]. Similarly the moderate Markov left-continuous dual process $(\widehat{X}_t,\widehat{\mathbf{P}}^x)$ associated with X and m is as described on p. 106 of [G99]. See also [Ma93] or [Fi87] where somewhat different notation is used. Let $\widehat{P}_t f = \widehat{\mathbf{P}}^\bullet[f(\widehat{X}_t)]$ and $\widehat{U}^\lambda f = \int_0^\infty e^{-\lambda t}\widehat{P}_t f\,dt = \widehat{\mathbf{P}}^\bullet \int_0^\infty e^{-\lambda t} f(\widehat{X}_t)\,dt$ denote the semigroup and resolvent of $\widehat{X}$. This semigroup and resolvent are linked to those of X by the duality formulas

$$(3.1)\quad (P_t f, g) = (f, \widehat{P}_t g) \quad\text{and}\quad (U^\lambda f, g) = (f, \widehat{U}^\lambda g), \qquad f,g\in\mathcal{E}^*, \lambda>0, t\ge 0,$$

in which $(f,g) := \int fg\,dm$ provided the integral exists. We emphasize that the probabilities $\widehat{\mathbf{P}}^x$, $x\in E$, are only uniquely determined off a Borel m-polar set (which may be taken to have absorbing complement); see [FG03, (5.14)]. Therefore functions involving the dual measures $\widehat{\mathbf{P}}^x$ are only well defined modulo an m-polar set. This should be borne in mind when reading formulas involving these functions. However this causes no problems with the duality formulas (3.1) since m does not charge m-polars, or even m-semipolars. Finally we shall usually omit the hat $\widehat{\ }$ in those places where it is obviously required. For example, we write $\widehat{\mathbf{P}}^\bullet[f(X_t)]$ in place of $\widehat{\mathbf{P}}^\bullet[f(\widehat{X}_t)]$. In most of the paper we shall make no use of the explicit realizations of X and $\widehat{X}$. However we shall need them in the Appendix. As mentioned in the introduction the reader may avoid the use of the moderate Markov dual by assuming throughout that X and $\widehat{X}$ are standard processes in weak duality with respect to m.

Because $b\in E$ is regular, we have $\mathbf{P}^b[T_b=0]=1$. By (A.4), $\widehat{\mathbf{P}}^b[T_b=0]=1$ as well. The following "hatted" forms of notation introduced in section 2 will be used freely in what follows: $\widehat{\varphi} := \widehat{\mathbf{P}}^{\bullet}[T_b < \infty]$, $\widehat{X}^b := (\widehat{X}, \widehat{T}_b)$ (the process $\widehat{X}$ killed at $\widehat{T}_b$), $\widehat{V}^\lambda$ and $\widehat{Q}_t$ (the resolvent and semigroup of $\widehat{X}^b$), etc. According to (A.7), X^b and $\widehat{X}^b$ are dual processes in the sense that $(V^\lambda f, g) = (f, \widehat{V}^\lambda g)$ for $f, g \in p\,\mathcal{E}^*$. See also Remark (A.8). As with their dual counterparts, we have $\widehat{R} = \widehat{T}_b$ on $\{\widehat{T}_b < \infty\}$. Here one must be a little careful since the measures $(\widehat{\mathbf{P}}^x, x \in E)$ are only determined modulo an m-polar set. Thus this last statement holds $\widehat{\mathbf{P}}^x$-a.s. for $x \in E\backslash N$ where $N \in \mathcal{E}$ is m-polar for both X and $\widehat{X}$. We should emphasize, that X is the object of interest; $\widehat{X}$ is a convenient construct to help us analyze X.

The Revuz measure ν_ℓ^m of ℓ with respect to m is proportional to ε_b, the unit mass at b; that is $\nu_\ell^m = c(m)\varepsilon_b$, where $0 < c(m) < \infty$. The constant $c(m)$ will appear in various formulas below. Of course ε_b is a smooth measure. Let $\widehat{\ell}$ be the dual PCAF of $\widehat{X}$ with Revuz measure $c(m)\varepsilon_b$. The λ-potential function

$$\hat{u}^\lambda(x) := \widehat{\mathbf{P}}^x \int_0^\infty e^{-\lambda t}\, d\ell_t \tag{3.2}$$

is only determined off an m-polar set, but since $\{b\}$ is not even semipolar, the value $\hat{u}^\lambda(b)$ is uniquely determined. Moreover the dual of (2.2) holds:

$$\widehat{U}_\ell^\lambda f(x) = \widehat{\varphi}^\lambda(x) \cdot \widehat{U}_\ell^\lambda f(b) \quad \text{and} \quad \widehat{u}^\lambda = \widehat{\varphi}^\lambda \cdot \widehat{u}^\lambda(b), \tag{3.3}$$

but requires a separate proof which is given in the Appendix as (A.9). From (A.2) we obtain the important relation

$$u^\lambda(b) = \int_E U_\ell^\lambda 1(x)\, \varepsilon_b(dx) = \int_E \widehat{U}_\ell^\lambda 1(x)\, \varepsilon_b(dx) = \hat{u}^\lambda(b). \tag{3.4}$$

Recall the "stickiness" coefficient γ from (2.8) and (2.9). Taking $\nu = m$ and $\mu = c(m)\varepsilon_b$ in (A.1) gives

$$\lambda m U^\lambda 1_b = \lambda(1, \gamma\, u^\lambda) = c(m)\gamma\lambda\widehat{U}^\lambda 1(b) \to c(m)\gamma \text{ as } \lambda \to \infty.$$

But $\lambda m U^\lambda 1_b \uparrow m(\{b\})$ as $\lambda \to \infty$ and so

$$\gamma = m(\{b\})/c(m). \tag{3.5}$$

We come now to the main result of this section. Recall from (2.7) the Lévy exponent $g(\lambda)$.

(3.6) Theorem. *If $f \in p\mathcal{E}^*$ and $\lambda > 0$, then*

(i) $g(\lambda) = \delta + \lambda(\widehat{\varphi}, \varphi^\lambda)/c(m)$, *and*

(ii) $U^\lambda f(b) = \dfrac{(f, \widehat{\varphi}^\lambda)/c(m)}{g(\lambda)}$.

Proof. It suffices to consider strictly positive $f \in b\mathcal{E}$ such that $m(f) < \infty$. Fix $\lambda > 0$ and $\beta \in]0, \lambda[$. From the resolvent equation $U^\lambda f(b) + (\lambda - \beta) U^\beta U^\lambda f(b) = U^\beta f(b)$. Then (A.1) with $\nu = c(m)\varepsilon_b$ and $\mu = m$ (*i.e.*, $A_t = (t \wedge \zeta)$) implies that $c(m)U^\lambda f(b) + (\lambda - \beta) \cdot (U^\lambda f, \hat{u}^\beta) = (f, \hat{u}^\beta)$. But from (3.3) and (3.4), $\hat{u}^\beta = \hat{u}^\beta(b)\widehat{\varphi}^\beta = u^\beta(b)\widehat{\varphi}^\beta$. Divide the previous equality by $u^\beta(b) \in]0, \infty[$ and then let $\beta \downarrow 0$. Since $u^\beta(b) = g(\beta)^{-1}$, we deduce from (2.2) and (2.8) that

$$c(m)\delta \cdot U^\lambda f(b) + \lambda(V^\lambda f, \widehat{\varphi}) + \lambda U^\lambda f(b) \cdot (\varphi^\lambda, \widehat{\varphi}) = (f, \widehat{\varphi}). \tag{3.7}$$

Because $f > 0$ we have $(\varphi^\lambda, \widehat{\varphi}) < \infty$, and then rearranging (3.7) we obtain

$$U^\lambda f(b)[c(m)\delta + \lambda(\varphi^\lambda, \widehat{\varphi})] = (f, \widehat{\varphi} - \lambda \widehat{V}^\lambda \widehat{\varphi}).$$

Using the moderate Markov property and the fact that $\{t \le \widehat{R}\} \in \widehat{\mathcal{F}}_{t-}$ one readily checks that $\widehat{\varphi} - \lambda \widehat{V}^\lambda \widehat{\varphi} = \widehat{\varphi}^\lambda$, which shows that

$$U^\lambda f(b) = \frac{(f, \widehat{\varphi}^\lambda)}{c(m)\delta + \lambda(\varphi^\lambda, \widehat{\varphi})} \tag{3.8}$$

in view of the last display. Finally, by (A.1) again, we see that $c(m)U^\lambda f(b) = (f, \hat{u}^\lambda) = \hat{u}^\lambda(b)(f, \widehat{\varphi}^\lambda) = u^\lambda(b)(f, \widehat{\varphi}^\lambda)$. Thus (3.8) implies that $g(\lambda) = u^\lambda(b)^{-1}$ $= \delta + \lambda(\varphi^\lambda, \widehat{\varphi})/c(m)$. This proves both (i) and (ii). $\square$

Taken together with its dual, (3.6)(ii) yields the following result.

(3.9) Proposition. $(\varphi^\lambda, \widehat{\varphi}) = (\varphi, \widehat{\varphi}^\lambda)$.

Proof. From (A.11), $\widehat{U}^\lambda f = \widehat{V}^\lambda f + \widehat{\varphi}^\lambda \cdot \widehat{U}^\lambda f(b)$ for $\lambda > 0$ and $f \in pb\mathcal{E}^*$. Now arguing as in the proof (3.6)(ii) we find that $\widehat{U}^\lambda f(b)[c(m)\delta + \lambda(\widehat{\varphi}^\lambda, \varphi)] = (f, \varphi^\lambda)$ since $\hat{u}^\beta(b) = u^\beta(b) \uparrow \delta^{-1}$ as $\beta \downarrow 0$. But $\widehat{U}^\lambda f(b) = c(m)^{-1}(f, u^\lambda) = c(m)^{-1}u^\lambda(b)(f, \varphi^\lambda)$ and so $g(\lambda) = u^\lambda(b)^{-1} = \delta + \lambda(\widehat{\varphi}^\lambda, \varphi)/c(m)$. Combining this with (3.6)(i) we obtain $(\varphi^\lambda, \widehat{\varphi}) = (\widehat{\varphi}^\lambda, \varphi)$. $\square$

(3.10) Remark. Since $(\varphi^\lambda, \widehat{\varphi}) = c(m)u^\lambda(b)^{-1}\widehat{U}^\lambda \widehat{\varphi}(b)$ and $(\widehat{\varphi}^\lambda, \varphi) = c(m)\hat{u}^\lambda(b)^{-1}U^\lambda \varphi(b)$ and $u^\lambda(b) = \hat{u}^\lambda(b)$, we see that (3.9) is equivalent to $U^\lambda \varphi(b) = \widehat{U}^\lambda \widehat{\varphi}(b)$.

We are now going to develop a formula that relates δ to the energy functional L^b of the killed process $X^b = (X, T_b)$. We refer the reader to [G90, §3] for general information about the energy functional. A σ-finite measure ξ is an X^b-excessive measure provided $\lambda\xi V^\lambda \le \xi$ on $E_b = E\backslash\{b\}$, which is the state space of X^b, since $\mathbf{P}^x[R > 0] = 1$ if $x \ne b$. If ξ is an X^b-excessive measure and f an X^b-excessive function, then

$$L^b(\xi, f) := \sup\{\mu(f) : \mu V \le \xi\}. \tag{3.11}$$

If ξ is purely excessive for X^b (that is $\xi Q_t(B) \downarrow 0$ as $t \to \infty$ whenever $B \in \mathcal{E}$ and $\xi(B) < \infty$) then (3.6) of [G90] states that

$$L^b(\xi, f) =\uparrow \lim_{\lambda\to\infty} \lambda\langle \xi - \lambda\xi V^\lambda, f\rangle \tag{3.12}$$

Define $\psi := 1 - \varphi = \mathbf{P}^\bullet[T_b = \infty]$. Since $Q_t\psi = \mathbf{P}^\bullet[T_b = \infty, t < R] \uparrow \psi$ on E_b as $t \downarrow 0$, the function ψ is X^b-excessive. (More precisely, the restriction of ψ to E_b is X^b-excessive; note that $\psi(b) = 0$.) Now

$$\widehat{Q}_t\widehat{\varphi} = \widehat{\mathbf{P}}^\bullet[t < \widehat{T}_b < \infty] \le \widehat{\varphi},$$

and in particular $\widehat{Q}_t\widehat{\varphi} \to 0$ as $t \to \infty$. Let $m_0 := \widehat{\varphi}m\big|_{E_b}$. If $f \in p\mathcal{E}$ with $m(f) < \infty$ and $f(b) = 0$, then $m_0Q_t(f) = (\widehat{\varphi}, Q_tf) = (\widehat{Q}_t\widehat{\varphi}, f) \le m_0(f)$ and $m_0Q_t(f) \to 0$ as $t \to \infty$. Therefore $m_0 = \widehat{\varphi}m\big|_{E_b}$ is a purely excessive measure for X^b, and so from (3.12)

$$L^b(m_0, \psi) =\uparrow \lim_{\lambda\to\infty} \lambda\langle m_0 - \lambda m_0 V^\lambda, \psi\rangle =\uparrow \lim_{\lambda\to\infty} \lambda(\widehat{\varphi} - \lambda\widehat{V}^\lambda\widehat{\varphi}, \psi). \tag{3.13}$$

The last equality is valid because $\psi(b) = 0$ and $m_0 - \lambda m_0V^\lambda = (\widehat{\varphi} - \lambda\widehat{V}^\lambda\widehat{\varphi})m$ as σ-finite measures on E_b. (In general, it is not the case that $\langle m_0 - \lambda m_0V^\lambda, \psi\rangle = (\widehat{\varphi}, \psi) - \lambda(\widehat{V}^\lambda\widehat{\varphi}, \psi)$.) Since $\widehat{\varphi}^\lambda = \widehat{\varphi} - \lambda\widehat{V}^\lambda\widehat{\varphi}$, it follows that

$$L^b(m_0, \psi) =\uparrow \lim_{\lambda\to\infty} \lambda(\widehat{\varphi}^\lambda, \psi). \tag{3.14}$$

We are now prepared to state the promised result relating δ and L^b. Recall from (2.20) that $\delta = \mathbf{P}^*[T_b = \infty]$.

(3.15) Theorem.

$$\begin{aligned} \delta &= c(m)^{-1}L^b(m_0, \psi) + \lim_{\lambda\to\infty} g(\lambda)[1 - \lambda U^\lambda 1(b)] \\ &= c(m)^{-1}L^b(m_0, \psi) + \mathbf{P}^*[\zeta = 0]. \end{aligned} \tag{3.16}$$

Consequently,

$$L^b(m_0, \psi) = c(m)\mathbf{P}^*[\zeta > 0, T_b = \infty]. \tag{3.17}$$

Proof. From (3.8) and (3.9)

$$U^\lambda 1(b) = \frac{(\widehat{\varphi}^\lambda, 1)}{c(m)\delta + \lambda(\widehat{\varphi}^\lambda, \varphi)} = \frac{(\widehat{\varphi}^\lambda, 1)}{c(m)\delta - \lambda(\widehat{\varphi}^\lambda, \psi) + \lambda(\widehat{\varphi}^\lambda, 1)}. \tag{3.18}$$

But $\lambda U^\lambda 1(b) \le 1$ and $g(\lambda) = \delta + \lambda(\widehat{\varphi}^\lambda, \varphi)/c(m) < \infty$ which forces $(\widehat{\varphi}^\lambda, 1) < \infty$. Now $(\widehat{\varphi}^\lambda, 1) = \hat{u}^\lambda(b)^{-1}(\hat{u}^\lambda, 1) = g(\lambda)c(m)U^\lambda 1(b)$. Substituting this into the last display we find

$$\delta = \lambda(\widehat{\varphi}^\lambda, \psi)/c(m) + g(\lambda)[1 - \lambda U^\lambda 1(b)]. \tag{3.19}$$

Letting $\lambda \to \infty$ establishes the first equality in (3.16). To see the second notice that on the event $\{\zeta < \infty\}$, we have $z = \zeta(\omega)$ if and only if $z = s + \zeta(\theta_s\omega)$ where s is the unique element of $G(\omega)$ for which $T_b(\theta_s\omega) = \infty$, and so

$$\begin{aligned} 1 - \lambda U^\lambda 1(b) &= \mathbf{P}^b[e^{-\lambda\zeta}] \\ &= \mathbf{P}^b \sum_{s \in G} e^{-\lambda s} e^{-\lambda\zeta\circ\theta_s} 1_{\{T_b = \infty\}}\circ\theta_s \\ &= \mathbf{P}^*[e^{-\lambda\zeta}; T_b = \infty] \cdot u^\lambda(b). \end{aligned}$$

Therefore

$$g(\lambda)[1 - \lambda U^\lambda 1(b)] = \mathbf{P}^*[e^{-\lambda\zeta}; T_b = \infty]. \tag{3.20}$$

The right side of (3.20) decreases to $\mathbf{P}^*[\zeta = 0, T_b = \infty] = \mathbf{P}^*[\zeta = 0]$ as $\lambda \to \infty$, and this gives the second equality in (3.16) $\square$

(3.21) Remarks. (a) Theorem (3.15) tells us that $L^b(m_0, \psi) \le c(m)\delta < \infty$. Since $\lambda^{-1}g(\lambda) \to \gamma < \infty$ as $\lambda \to \infty$, a sufficient condition that $\delta = L^b(m_0, \psi)/c(m)$ is that $\lambda[1 - \lambda U^\lambda 1(b)] \to 0$ as $\lambda \to \infty$, and this condition is necessary if $\gamma > 0$. If $\gamma = 0$, then the condition $\limsup_{\lambda\to\infty} \lambda[1 - \lambda U^\lambda 1(b)] < \infty$ suffices.

(b) The formula (2.20) identifies δ as the (local time) rate at which a final (infinite duration) excursion appears, thereby terminating the visiting set $\{t : X_t = b\}$. Formulas (3.16) and (3.17) indicate that $\mathbf{P}^*[\zeta = 0] \le \delta$ is the rate at which the process X is killed at b, while $L^b(m_0, \psi)/c(m)$ is the rate of appearance of an excursion in which the process wanders away from b, never to return. See Remark (2.21)(a). In particular, $\delta = c(m)^{-1}L^b(m_0, \psi)$ if and only if there is no killing at b.

It is easy to check that the measure $\widehat{\varphi}m$ is excessive for X. Indeed, $\widehat{\varphi}m$ is the balayage of m on the singleton $\{b\}$; see Proposition (A.3) and the discussion preceding it in the Appendix. Let m_b denote the restriction of m to E_b. Then

$$V_*^\lambda = c(m)^{-1}\widehat{\varphi}^\lambda m_b, \quad \lambda \geq 0, \tag{3.22}$$

by (2.16) and (3.6).

(3.23) Corollary. *The entrance law $(Q_t^*)_{t>0}$ is uniquely determined by the measure $c(m)^{-1}m_0$.*

Proof. Note that $\widehat{\varphi}\, m_b = m_0 = \widehat{\varphi}\, m|_{E_b}$ is a purely excessive measure for (Q_t). See the discussion just before (3.13). Moreover $m_0 = c(m)V_* = c(m)\int_0^\infty Q_t^* dt$. It is well-known that a purely excessive measure is the integral of a uniquely determined entrance law. See, *e.g.*, [G90, (5.25)]. This establishes the corollary. □

(3.24) Remark. If $\hat{\zeta} = \infty$ so that $\widehat{T}_b = \widehat{R}$, then $c(m)^{-1}m_0 (= c(m)^{-1}\widehat{\varphi}\, m_b)$ and the entrance law (Q_t^*) are uniquely determined by the killed process $\widehat{X}^b$.

We conclude this section with a brief discussion of the resolvent decomposition resulting from (2.2), (2.16), (3.8), and (3.9):

$$\begin{aligned} U^\lambda f(x) &= V^\lambda f(x) + \varphi^\lambda(x)\,\frac{(\widehat{\varphi}^\lambda, f)}{c(m)\delta + \lambda(\widehat{\varphi}^\lambda, \varphi)} \\ &= V^\lambda f(x) + \varphi^\lambda(x)\,\frac{\gamma f(b) + V_*^\lambda(f)}{\delta + \gamma\lambda + \lambda V_*^\lambda(\varphi)}, \qquad x \in E, \end{aligned} \tag{3.25}$$

vis-à-vis the killed process X^b. Recalling that R (the lifetime of X^b) is equal to $T_b \wedge \zeta$, we see that $\{T_b < \infty\} = \{R < \infty, X_R = b\}$. Therefore $\varphi^\lambda = \mathbf{P}^\bullet[e^{-\lambda R}; X_R = b]$ and $\varphi = \mathbf{P}^\bullet[X_R = b]$. Because the entrance law $(Q_t^*)_{t>0}$ is determined by $c(m)^{-1}\widehat{\varphi}m_b$, so is its Laplace transform $(V_*^\lambda)_{\lambda \geq 0}$. Also, $\gamma = \lim_{\lambda\to\infty} g(\lambda)/\lambda = \lim_{\lambda\to\infty}(\varphi^\lambda, \widehat{\varphi})/c(m)$. Consequently, all quantities appearing on the right side of (3.25) (except δ) are determined by the *stopped* processes $X_{t\wedge R}$ and $\widehat{X}_{t\wedge R}$. If both ζ and $\widehat{\zeta}$ are infinite (so that $R = T_b$ and $\widehat{R} = \widehat{T}_b$), then the resolvent (U^λ) is uniquely determined by the *killed* processes since (3.16) implies that $\delta = L^b(m_0, \psi)/c(m)$ in this case.

Finally, we relate the parameters of the decomposition (3.25) to the resolvent decomposition [R83, (7)], obtained there by purely analytic arguments.

We remark at the outset that there appears to be an unspoken assumption in [R83], which when expressed in our notation amounts to this: $\mathbf{P}^x[T_b > \zeta] = 0$ for all $x \in E$—in other words, the process X can be killed only when in state b. For example, implicit in formula (6) of [R83] is the identity of $\mathbf{P}^\bullet[e^{-\lambda R}]$ and $\mathbf{P}^\bullet[e^{-\lambda T_b}]$ (our notation). This condition implies that $\mathbf{P}^x[\zeta < \infty, T_b = \infty] = 0$ for all $x \in E$, and in turn that

$$
\begin{aligned}
\mathbf{P}^*[0 < \zeta < \infty; T_b = \infty] &= \lim_{t\downarrow 0} \mathbf{P}^*[\theta_t^{-1}\{\zeta < \infty, T_b = \infty\}; t < \zeta] \\
&= \lim_{t\downarrow 0} \int_{E_b} \mathbf{P}^x[\zeta < \infty, T_b = \infty]\, Q_t^*(dx) = 0.
\end{aligned}
\tag{3.26}
$$

Getting to the point, Rogers' decomposition is

$$
U^\lambda f(b) = \frac{\gamma_0 f(b) + n_\lambda(f)}{\delta_0 + \gamma_0\lambda + \lambda n_\lambda(1_{E_b})},
\tag{3.27}
$$

where $\gamma_0 \geq 0$, $\delta_0 \geq 0$, and $(n_\lambda)_{\lambda>0}$ is the Laplace transform of an entrance law for (Q_t). (We have written γ_0 and δ_0 for what Rogers calls γ and δ, so as to distinguish his coefficients from ours.) To streamline the discussion, we suppose for the remainder of the section that the local time ℓ has been normalized so that $c(m) = 1$. We start with the expression [R83, (9)] for the numerator on the right side of (3.27):

$$
\gamma_0 f(b) + n_\lambda(f) = U^\lambda f(b) \cdot \left[1 + (\lambda - \beta) U^\beta \varphi^\lambda(b)\right]
$$

(here we use the assumption that $\varphi^\lambda = \mathbf{P}^\bullet[e^{-\lambda R}]$). Our computation relies on the following identity, in which $D_t := t + T_b \circ \theta_t$:

$$
\begin{aligned}
\mathbf{P}^b\left[\int_t^\infty e^{-\lambda s}\, d\ell_s\right] &= \mathbf{P}^b \int_{D_t}^\infty e^{-\lambda s}\, d\ell_s \\
&= \mathbf{P}^b\left[e^{-\lambda D_t}\left(\int_0^\infty e^{-\lambda u}\, d\ell_u\right)\circ\theta_{D_t}\right] \\
&= \mathbf{P}^b[e^{-\lambda D_t}] u^\lambda(b).
\end{aligned}
$$

Using the above for the fourth equality below:

$$
\begin{aligned}
(\lambda-\beta)U^\beta\varphi^\lambda(b) &= (\lambda-\beta)\mathbf{P}^b\int_0^\infty e^{-\beta t}\mathbf{P}^{X_t}[e^{-\lambda T_b}]\,dt \\
&= (\lambda-\beta)\mathbf{P}^b\int_0^\infty e^{-\beta t}e^{-\lambda T_b\circ\theta_t}\,dt \\
&= (\lambda-\beta)\int_0^\infty e^{-(\beta-\lambda)t}\mathbf{P}^b[e^{-\lambda(t+T_b\circ\theta_t)}]\,dt \\
&= g(\lambda)(\lambda-\beta)\int_0^\infty e^{-(\beta-\lambda)t}\mathbf{P}^b\left[\int_t^\infty e^{-\lambda s}\,d\ell_s\right]\,dt \\
&= g(\lambda)\mathbf{P}^b\left[\int_0^\infty\left(\int_0^s(\lambda-\beta)e^{-(\beta-\lambda)t}\,dt\right)e^{-\lambda s}\,d\ell_s\right] \\
&= g(\lambda)\mathbf{P}^b\left[\int_0^\infty(e^{-\beta t}-e^{-\lambda t})\,d\ell_s\right] \\
&= g(\lambda)\left[g(\beta)^{-1}-g(\lambda)^{-1}\right].
\end{aligned}
$$

It follows that

$$\gamma_0 f(b)+n_\lambda(f)=U^\lambda f(b)\cdot\frac{g(\lambda)}{g(\beta)}. \tag{3.28}$$

Using (3.28) to compare (3.27) with (3.25) we find that

$$\gamma_0=g(\beta)^{-1}\gamma \quad\text{and}\quad n_\lambda=g(\beta)^{-1}V_*^\lambda. \tag{3.29}$$

To compute δ_0 we recall the definition $\delta_0:=1-\beta U^\beta 1(b)$, where $\beta>0$ is fixed (but arbitrary). Therefore, by a computation appearing in the display above (3.20),

$$\delta_0=\mathbf{P}^b[e^{-\beta\zeta}]=g(\beta)^{-1}\mathbf{P}^*[\zeta=0], \tag{3.30}$$

because of (3.26). This reconciles the discussion preceding (3.26) with the observation in Remark (3.21)(b) that $\mathbf{P}^*[\zeta=0]$ is the rate of killing at b.

4. Excessive Measures from Excursions

We continue in the setting of section 3. It follows from (3.22) and (3.5) that

$$\widehat{\varphi}m=c(m)\left[\gamma\varepsilon_b+V_*\right]. \tag{4.1}$$

The measure $\widehat{\varphi}m$ is X-excessive (as noted in section 3, it is the balayage of m on the singleton $\{b\}$; as such the identity (4.1) has been noted already in [FM86, (6.10)].) According to [G79, Thm. 8.1], the excessive measure $\gamma\varepsilon_b + V_*$ is an invariant measure for (P_t) if and only if $\delta = 0$, at least under the side condition $\zeta = \infty$ a.s. (The coefficient δ is denoted by $h(\infty)$ in [G79].) We shall give a simpler proof of a sharper form of this assertion without assuming the finiteness of ζ. Our proof relies on the following lemma. Recall that L is the last exit time from the state b, and let $\widehat{L} := \sup\{t \geq 0 : \widehat{X}_t = b\}$ denote the dual object.

(4.2) Lemma. *The distribution of L under $\mathbf{P}^b$ is the same as the distribution of $\widehat{L}$ under $\widehat{\mathbf{P}}^b$.*

Proof. As noted already in the proof of (2.22),

$$\mathbf{P}^b[e^{-\lambda L}] = 1 - \lambda U^\lambda \varphi(b),$$

and by the same token

$$\widehat{\mathbf{P}}^b[e^{-\lambda L}] = 1 - \lambda \widehat{U}^\lambda \widehat{\varphi}(b).$$

But $\widehat{U}^\lambda \widehat{\varphi}(b) = U^\lambda \varphi(b)$—see the remark following the proof of (3.9)—and so $\mathbf{P}^b[e^{-\lambda L}] = \widehat{\mathbf{P}}^b[e^{-\lambda L}]$ for all $\lambda > 0$. $\square$

(4.3) Proposition. *The excessive measure $\widehat{\varphi}m$ is invariant for (P_t) if $\delta = 0$, otherwise it is purely excessive.*

Proof. Evidently $\widehat{\varphi}\, m$ is invariant (resp. purely excessive) for (P_t) if and only if $\widehat{P}_t \widehat{\varphi} = \widehat{\varphi}$, m-a.e. for all $t > 0$ (resp. $\lim_{s\to\infty} \widehat{P}_s \widehat{\varphi} = 0$, m-a.e.). Define $\hat{\sigma} :=\downarrow \lim_{s\to\infty} \widehat{P}_s \widehat{\varphi}$. Then $\hat{\sigma}$ is invariant for $(\widehat{P}_t)$. Since $\{\widehat{T}_b \circ \widehat{\theta}_t < \infty\} = \{\widehat{L} > t\}, \hat{\sigma} = \lim_{s\to\infty} \widehat{\mathbf{P}}^\bullet[L > s] = \widehat{\mathbf{P}}^\bullet[L = \infty]$. Now $\widehat{T}_b < \infty$ on $\{\widehat{L} = \infty\}$ and so using the moderate Markov property at the $\widehat{X}$-predictable time $\widehat{T}_b + \varepsilon$ ($\varepsilon > 0$), we have

$$\hat{\sigma} = \widehat{\mathbf{P}}^\bullet[L = \infty, T_b + \varepsilon < \infty] = \widehat{\mathbf{P}}^\bullet[\hat{\sigma}(X_{T_b+\varepsilon}) : T_b < \infty].$$

Upon letting $\varepsilon \downarrow 0$ we see as in the proof of (A.9) that $\hat{\sigma} = \widehat{\mathbf{P}}^b[L = \infty]\widehat{\varphi}$, m-a.e. But (4.2) tells us that $\widehat{\mathbf{P}}^b[L = \infty] = \mathbf{P}^b[L = \infty]$, and (by the final assertion of (2.17)) $\mathbf{P}^b[L < \infty]$ is equal to 0 or 1 according as $\delta = 0$ or $\delta > 0$. Thus $\hat{\sigma} = \widehat{\varphi}$ and hence $\widehat{\varphi}m$ is (P_t) invariant if and only if $\delta = 0$, and $\widehat{\sigma} = 0$, m-a.e. if $\delta > 0$. $\square$

There is a classical construction, going back to T. Harris [H56] in the context of Markov chains, of an invariant measure as the mean occupation measure of an excursion. The same construction, in the context of right processes, has been discussed by Getoor [G79, §8] (who credits the result to M. Silverstein). The idea is simple: notice that the right side of (4.1) depends on the excessive measure m only through the constant $c(m)$. So let us forget about m and use the right side of (4.1) to *define* a measure on $(E, \mathcal{E})$:

$$\xi := \gamma\varepsilon_b + V_*, \tag{4.4}$$

where $(\ell, \mathbf{P}^*)$ is an exit system for the excursions from b, as described in section 2. In fact this measure is a special instance of a general construction found in [FG88, §5], which is concerned with inverting the map $m \mapsto \nu_A^m$ from excessive measure m to Revuz measure ν_A^m of a general PCAF A of X. The following result summarizes things in our situation.

(4.5) Theorem. *Let the measure ξ be defined as in (4.4). Then*
(i) $\xi P_t \le \xi$ for all $t > 0$;
(ii) ξ is σ-finite on $\{\varphi > 0\}$;
(iii) ξ is X-excessive if and only V_* is σ-finite on E_b.

Let us now suppose that ξ is σ-finite, hence X-excessive. Then by [FG88, (5.11)], the balayage of ξ on $\{b\}$ is equal to ξ. In other words,

$$\widehat{\varphi}_\xi \cdot \xi = \xi, \tag{4.6}$$

where $\widehat{\varphi}_\xi(x) := \widehat{\mathbf{P}}_\xi^x[T_b < \infty]$, and $\widehat{\mathbf{P}}_\xi^x$ is the law of the moderate Markov dual process (started at x) when the duality measure is taken to be ξ. Thus, $\widehat{\varphi}_\xi = 1$, ξ-a.e. Moreover, ξ is a (P_t)-invariant measure if and only if $\delta = 0$.

5. Examples

In this section we present several simple examples illustrating some of the results of Sections 3 and 4. To simplify things we normalize the local time ℓ so that $c(m) = 1$.

(5.1) Example. Let $B = (B_t)$ be Brownian motion on $\mathbf{R}$ and define $X_t = B_t + \mu t$ where $\mu > 0$. Thus $X = (X_t)$ is Brownian motion with constant drift μ to the right. Then X and $\widehat{X}_t = B_t - \mu t$ are in classical duality with respect to Lebesgue measure, m, on $\mathbf{R}$. Let $b = 0, T = T_0$ and $\beta = (\mu^2 + 2\lambda)^{1/2}$. Then one easily calculates that

$$\varphi^\lambda(x) = \begin{cases} e^{-(\mu+\beta)x}, & x > 0, \\ e^{-(\mu-\beta)x}, & x \le 0, \end{cases}$$

and

$$\varphi(x) = \begin{cases} e^{-2\mu x}, & x > 0, \\ 1, & x \le 0. \end{cases}$$

The dual object $\widehat{\varphi}^\lambda$ is obtained by replacing μ by $-\mu$ when $\lambda > 0$, and

$$\widehat{\varphi}(x) = \begin{cases} 1, & x > 0, \\ e^{2\mu x}, & x \le 0. \end{cases}$$

Thus one finds

$$(\widehat{\varphi}^\lambda, \varphi) = \frac{2}{\mu + \beta} = (\varphi^\lambda, \widehat{\varphi}). \tag{5.2}$$

Moreover $(\widehat{\varphi}^\lambda, \psi) = (\widehat{\varphi}^\lambda, 1 - \varphi) = \mu/\lambda$. Consequently from (3.6), (3.14), and (3.21), $L^0(m_0, \psi) = \mu > 0$ and

$$g(\lambda) = \mu + \frac{2\lambda}{\mu + (\mu^2 + 2\lambda)^{1/2}}. \tag{5.3}$$

This reduces to $g(\lambda) = \sqrt{2\lambda}$ when $\mu = 0$, the familiar exponent for the inverse local time for Brownian motion. The entrance law (Q_t^*) is also easily found. Since the Laplace transform of $Q_t^*(f)$ is $(\widehat{\varphi}^\lambda, 1_{E_0} f)$ by (3.22), inverting the Laplace transform one finds that $Q_t^*(dx) = q(t,x)dx$, where

$$q(t,x) = \frac{|x|}{\sqrt{2\pi t^3}} e^{\mu x} e^{-\mu^2 t/2} e^{-x^2/2t}. \tag{5.4}$$

Again this reduces to a well-known entrance law for Brownian motion when $\mu = 0$.

(5.5) Example. In the resolvent decomposition

$$U^\lambda f(b) = \frac{\gamma f(b) + V_*^\lambda(f)}{\delta + \gamma\lambda + \lambda V_*^\lambda(\varphi)}, \tag{5.6}$$

the ingredients $\varphi = \mathbf{P}^\bullet[T_b < \infty] = \mathbf{P}^\bullet[X_R = b]$ and $V_*^\lambda = \int_0^\infty e^{-\lambda t} Q_t^*\, dt$ are determined by the stopped process $(X_{t\wedge R})_{t\ge 0}$ and the entrance law $(Q_t^*)_{t>0}$. In this example and the next we examine the extent to which γ and δ are free parameters.

Fix $\tilde{\gamma} \ge 0$ and define a PCAF of X by the formula

$$A_t = (t \wedge \zeta) + (\tilde{\gamma} - \gamma)\ell_t, \qquad t \ge 0. \tag{5.7}$$

That A_t is increasing follows from (2.9): If $0 \le s \le t$,

$$A_t - A_s \ge \int_s^t 1_b(X_s)\, ds + (\tilde\gamma - \gamma)(\ell_t - \ell_s) = \tilde\gamma(\ell_t - \ell_s). \tag{5.8}$$

Let us now use A to time change X. Thus define

$$\rho(t) := \inf\{s > 0 : A_s > t\}, \qquad t \ge 0, \tag{5.9}$$

and

$$\tilde X_t := X_{\rho(t)}, \qquad t \ge 0. \tag{5.10}$$

It is standard that $\tilde X$ is a right Markov process. The PCAF A is strictly increasing if $\tilde\gamma > 0$, and even if $\tilde\gamma = 0$ provided b is not a holding point. In either of these cases the point b is regular for $\tilde X$, with local time at b given naturally by

$$\tilde\ell_t := \ell_{\rho(t)}. \tag{5.11}$$

Observe that

$$\int_0^t 1_b(\tilde X_s)\, ds = \int_0^{\rho(t)} 1_b(X_u)\, dA_u = \tilde\gamma \tilde\ell_t. \tag{5.12}$$

Since the time change has no effect on X during excursions from b, it is not surprising (and not difficult to verify) that the resolvent of $\tilde X$ is given by

$$\tilde U^\lambda f(x) = V^\lambda f(x) + \varphi^\lambda(x) \frac{\tilde\gamma f(b) + V_*^\lambda(f)}{\delta + \tilde\gamma\lambda + \lambda V_*^\lambda(\varphi)} \tag{5.13}$$

In short, by making a suitable time change one can alter the "stickiness" parameter γ to be any non-negative real $\tilde\gamma$ (with the proviso that $\tilde\gamma > 0$ if b is a holding point), while leaving everything else unchanged.

(5.14) Example. We now consider alteration of the parameter δ in (5.6). To this end we employ the resurrection procedure discussed in [Me75]. Run the process X until it dies. If $\zeta = L$, then restart the process in state b at that time and continue until the next death, if any. At the second death resurrect the process again, but only if $\zeta^{(2)} = L^{(2)}$ (where $\zeta^{(2)}$ denote the additional lifetime due to the first resurrection and $L^{(2)}$ has the analogous meaning). Continue in this way forever, or until there occurs an excursion that dies away from b. Let us use $\overline{X}$ to denote the process so constructed; this is an instance of a general construction appearing in [Me75, §1], our "noyau de renaissance" being the kernel $N(\omega, dx) = 1_{\{\zeta=L\}}(\omega)\varepsilon_b(dx)$. We evidently have (employing the obvious notation)

$$\overline{P}_t f(x) = P_t f(x) + \int_{]0,t]} \mathbf{P}^x[\zeta \in ds; \zeta = L]\, \overline{P}_{t-s} f(b), \tag{5.15}$$

and so (by (2.19))

$$\overline{U}^\lambda f(x) = U^\lambda f(x) + \frac{\mathbf{P}^*[\zeta = 0]}{g(\lambda)}\, \overline{U}^\lambda f(b). \tag{5.16}$$

Therefore

$$\overline{U}^\lambda f(x) = V^\lambda f(x) + \varphi^\lambda(x)\, \overline{U}^\lambda f(b), \tag{5.17}$$

where

$$\begin{aligned} \overline{U}^\lambda f(b) &= U^\lambda f(b)/[1 - \mathbf{P}^*[\zeta = 0] g(\lambda)^{-1}] \\ &= \frac{\gamma f(b) + V_*^\lambda(f)}{g(\lambda)} \cdot \frac{g(\lambda)}{g(\lambda) - \mathbf{P}^*[\zeta = 0]} \\ &= \frac{\gamma f(b) + V_*^\lambda(f)}{\delta_{\min} + \gamma\lambda + \lambda V_*^\lambda(\varphi)}, \end{aligned} \tag{5.18}$$

and

$$\delta_{\min} := \delta - \mathbf{P}^*[\zeta = 0] = \mathbf{P}^*[\zeta > 0, T_b = \infty] = L^b(m_0, \psi), \tag{5.19}$$

in view of (3.16). In short, suppression of all killing at b replaces δ by $\delta_{\min}$.

We now show that any other $\tilde{\delta} \geq \delta_{\min}$ is possible in (5.6). To save on ink we suppose that the resurrection procedure of the preceding paragraph has been performed and that we are starting with a process satisfying $\delta = \delta_{\min}$. (In general, the discussion that follows must be applied to $\overline{X}$.) Suppose that $\tilde{\delta} > \delta_{\min}$. Define $k := \tilde{\delta} - \delta_{\min}$. Let us compute the resolvent (call it $(U^\lambda_{(k)})$) of the subprocess $X^{(k)} = (X, e^{-k\ell})$ corresponding to the multiplicative functional $e^{-k\ell_t}$. Using the fact that ℓ is constant on the excursion intervals from b, we have

$$\begin{aligned} &\mathbf{P}^b \int_0^\infty e^{-\lambda t} e^{-k\ell_t} f(X_t)\, dt \\ &\qquad = f(b)\mathbf{P}^b \int_0^\infty e^{-\lambda t} e^{-k\ell_t} \gamma\, d\ell_t \\ &\qquad\qquad + \mathbf{P}^b \sum_{s \in G} e^{-\lambda s} e^{-k\ell_s} \left(\int_0^R e^{-\lambda t} f(X_t)\, dt \right) \circ \theta_s \\ &\qquad = \left[\gamma f(b) + V_*^\lambda(f)\right] \cdot \mathbf{P}^b \int_0^\infty e^{-\lambda t} e^{-k\ell_t}\, d\ell_t. \end{aligned} \tag{5.20}$$

But

$$\begin{aligned}\mathbf{P}^b\int_0^\infty e^{-\lambda t}e^{-k\ell_t}\,d\ell_t &= \mathbf{P}^b\int_0^\infty e^{-\lambda\tau(u)}e^{-ku}\,du\\ &= \int_0^\infty e^{-ku-g(\lambda)u}\,du\\ &= [k+g(\lambda)]^{-1},\end{aligned}\tag{5.21}$$

and

$$k+g(\lambda) = k+\delta_{\min}+\gamma\lambda+\lambda V_*^\lambda(\varphi) = \tilde\delta+\gamma\lambda+\lambda V_*^\lambda(\varphi).\tag{5.22}$$

Therefore

$$U_{(k)}^\lambda f(b) = \frac{\gamma f(b)+V_*^\lambda(f)}{\tilde\delta+\gamma\lambda+\lambda V_*^\lambda(\varphi)}.\tag{5.23}$$

In summary, the parameter δ is free subject only to the constraint that it be at least $\delta_{\min}$ given in (5.19). As we noted just above (3.26), Rogers [R83] assumes that $\mathbf{P}^\bullet[T_b>\zeta]=0$; in view of (3.26) this means that in his context,

$$\delta_{\min} = \mathbf{P}^*[\zeta>0, T_b=\infty] = \mathbf{P}^*[\zeta=T_b=\infty],\tag{5.24}$$

which is consistent with the discussion at the end of section 3.

Appendix

In this appendix we shall prove the facts about the moderate Markov left-continuous dual process $\widehat{X}$ of our Borel right process that, to the best of our knowledge, are not readily available in the literature. Since this appendix is intended for those interested in Borel right processes we shall not repeat the basic definition of how $\widehat{X}$ is represented in terms of the stationary process Y and the Kuznetsov measure $\mathbf{Q}_m$ associated with X and m. A good summary of the relevant facts is contained in Section 2 of [G99]. However we should emphasize one point. The probabilities $\widehat{\mathbf{P}}^x$ are only uniquely determined off an m-polar set – actually off an m-exceptional set, see [FG03, (5.14)]. Therefore functions involving the dual measures $\widehat{\mathbf{P}}^x$ are only determined modulo m-polars. This causes no difficulties when such functions are integrated with respect to a measure that doesn't charge m-polars. We shall not mention this explicitly in what follows, but it should be kept in mind when reading formulas involving the dual process.

We remind the reader of the "one hat" convention discussed below (3.1).

Let μ be a smooth measure as in Section 3 of [FG96] and $\kappa = \kappa_\mu$ be the associated diffuse optional copredictable homogeneous random measure (HRM) of Y. Let A and $\widehat{A}$ be the corresponding positive continuous additive functionals of X and $\widehat{X}$ respectively. Then μ is the Revuz measure of both A and $\widehat{A}$. One may suppose that $A_t = \kappa]0,t]$ if $t < \zeta$ and $\widehat{A}_t = \kappa]-t,0[$ if $t < \hat{\zeta}$. Our first result extends (3.7) and (4.6) of [G99].

(A.1) Proposition. *Let μ and ν be smooth measures and let $A, \widehat{A}$ and $B, \widehat{B}$ correspond to μ and ν respectively. If $f, g \in p\mathcal{E}^*$ and $\lambda \geq 0$, then*

$$\int_E f \cdot U_A^\lambda g \; d\nu = \int_E g \cdot \widehat{U}_B^\lambda f \; d\mu$$

where $U_A^\lambda f = \mathbf{P}^\bullet \int_0^\infty e^{-\lambda t} f(X_t)\, dA_t$ and $\widehat{U}_B^\lambda f = \widehat{U}_{\widehat{B}}^\lambda f = \widehat{\mathbf{P}}^\bullet \int_0^\infty e^{-\lambda t} f(X_t)\, dB_t$

Proof. This result when one of the measures, say ν, is absolutely continuous with respect to m is contained in [G99, (4.6)] or, more generally, in [FG03, (5.12)]. In this case it takes the form $(f, U_A^\lambda g) = \int_E g \cdot \widehat{U}^\lambda f d\mu$, and dually $(f, \widehat{U}_A^\lambda g) = \int g \cdot U^\lambda f d\mu$. It suffices to consider $\lambda > 0$ and $f, g \in bp\mathcal{E}^*$; also, by replacing μ and ν by $g\mu$ and $f\nu$ we can reduce to the case $f = 1 = g$. Our task then is to show that $\nu(U_A^\lambda 1) = \mu(\widehat{U}_B^\lambda 1)$. Now $\beta \widehat{U}^{\beta+\lambda} \widehat{U}_B^\lambda 1 \uparrow \widehat{U}_B^\lambda 1$ as $\beta \to \infty$ since $\widehat{U}_B^\lambda 1$ is λ-coexcessive. Therefore using the special case discussed above,

$$\begin{aligned}\mu(\widehat{U}_B^\lambda 1) &= \lim_{\beta\to\infty} \beta\mu\left(\widehat{U}^{\beta+\lambda}\widehat{U}_B^\lambda 1\right)\\ &= \lim_{\beta\to\infty} \beta\left(\widehat{U}_B^\lambda 1, U_A^{\beta+\lambda} 1\right) = \lim_{\beta\to\infty} \beta\nu(U^\lambda U_A^{\beta+\lambda} 1).\end{aligned}$$

A simple calculation shows that $\beta U^\lambda U_A^{\beta+\lambda} 1 \leq U_A^\lambda 1$, and so $\mu(\widehat{U}_B^\lambda 1) \leq \nu(U_A^\lambda 1)$. A similar argument gives $\nu(U_A^\lambda 1) \leq \mu(\widehat{U}_B^\lambda 1)$, and this establishes (A.1). $\square$

Taking $\mu = \nu$ in (A.1) we have

$$\int_E f \cdot U_A^\lambda \, g \; d\mu = \int_E g \cdot \widehat{U}_A^\lambda f \, d\mu, \quad f, g \in p\mathcal{E}^*. \tag{A.2}$$

The following fact is needed in section 3 only for T_b, but is valid much more generally. Recall from [G90, §4] that the *balayage* $R_{\{b\}}m$ of the excessive measure m on the singleton $\{b\}$ is defined via the energy functionals L^λ, $\lambda > 0$, of X as

$$R_{\{b\}}m(f) := \uparrow \lim_{\lambda \downarrow 0} L^\lambda(m, P_{\{b\}}^\lambda U^\lambda f), \qquad f \in p\mathcal{E},$$

where $P_{\{b\}}^\lambda u := \mathbf{P}^\bullet[e^{-\lambda T_b} u(X_{T_b})] = \varphi^\lambda \cdot u(b)$. The measure $R_{\{b\}}m$ is excessive and is dominated by m.

(A.3) Proposition. *$R_{\{b\}}m = \widehat{\varphi}m$, where $\widehat{\varphi} = \mathbf{P}^\bullet[T_b < \infty]$.*

Proof. By [G90, (7.9)],

$$R_{\{b\}}m(f) = \mathbf{Q}_m[f(Y_0); \tau_b < 0],$$

where $\tau_b := \inf\{t : Y_t = b\}$. But on $\{\alpha < 0 < \beta\}$ we have $\{\tau_b < 0\} = \check{\theta}_0^{-1}\{\widehat{T}_b < \infty\}$, so

$$\mathbf{Q}_m[f(Y_0); \tau_b < 0] = \mathbf{Q}_m\left[f(Y_0)\widehat{\mathbf{P}}^{Y_0}[T_b < \infty]\right] = \mathbf{Q}_m[f(Y_0)\widehat{\varphi}(Y_0)] = m(f\widehat{\varphi}).$$

□

(A.4) Proposition. *If b is regular for X, then $\widehat{\mathbf{P}}^b[T_b = 0] = 1$.*

Proof. Let ℓ be the local time at b and κ the corresponding HRM normalized so that the Revuz measure of ℓ, or equivalently, of κ, is ε_b. Now κ is copredictable and κ doesn't charge α. If $h : \mathbf{R} \to]0, \infty[$ with $\int_{\mathbf{R}} h(t)dt = 1$, then

$$\begin{aligned}
\mathbf{Q}_m \int_{\mathbf{R}} h(t) 1_{\{\hat{T}_b \circ \check{\theta}_t = 0\}} \kappa(dt) &= \mathbf{Q}_m \int_{\mathbf{R}} h(t) \widehat{\mathbf{P}}^{Y(t)}[T_b = 0] \kappa(dt) \\
&= \widehat{\mathbf{P}}^b[T_b = 0]\varepsilon_b(1) \\
&= \widehat{\mathbf{P}}^b[T_b = 0],
\end{aligned}$$

where the second equality comes from [G90, (8.21)] and the fact that the HRM κ is carried by $\{t : Y_t = b\}$, $\mathbf{Q}_m$-a.e. On the other hand if $Z = \{t : Y_t = b\}^-$ where the "$^-$" denotes closure, $Z \cap \{t : \widehat{T}_b \circ \check{\theta}_t > 0\} = Z \cap \{t : \exists \varepsilon > 0,]t - \varepsilon, t[\cap Z = \phi\}$. Hence a $t \in Z \cap \{t : \widehat{T}_b \circ \check{\theta}_t > 0\}$ is the right endpoint of a maximal open interval in $\mathbf{R} \backslash Z$. But there are at most a countable number of such intervals and since κ is diffuse

$$\mathbf{Q}_m \int_{\mathbf{R}} h(t) 1_{\{\hat{T}_b \circ \check{\theta}_t = 0\}} \kappa(dt) = \varepsilon_b(1) = 1.$$

Consequently $\widehat{\mathbf{P}}^b[T_b = 0] = 1$. □

(A.5) Remark. In general Blumenthal's zero-one law does not hold for $\widehat{X}$. Hence $\widehat{\mathbf{P}}^b[T_b = 0] = 1$ is not equivalent to $\widehat{\mathbf{P}}^b[T_b = 0] > 0$. Since $\{b\}$ is not m-semipolar, $\widehat{\mathbf{P}}^b$ is uniquely determined.

The next result complements (A.4).

(A.6) Proposition. *If b is regular, then $\widehat{\mathbf{P}}^\bullet[\widehat{T}_b = 0] = 0$ on $E_b \backslash S$ where S is m-semipolar. Recall that $E_b = E\backslash\{b\}$.*

Proof. Let $\Gamma = \{x : d(x,b) > \varepsilon\}$, where d is a metric compatible with the topology of E and $\varepsilon > 0$. Let $g > 0$ with $m(g) < \infty$. Set $f = g1_\Gamma$ so that $f > 0$ on Γ and $m(f) < \infty$. Let μ be a finite measure not charging m-semipolar sets. Then a fundamental theorem of the first-named author states that there exists a diffuse optional copredictable HRM, κ, whose Revuz measure is μ. A proof may be fashioned from section 5, especially (5.22), of [Fi87]. See also (3.10) and (3.11) of [FG03] in this connection. As in the proof of (A.4),

$$\mathbf{Q}_m \int_{\mathbf{R}} h(t)\widehat{\mathbf{P}}^{Y(t)}[T_b = 0]f(Y_t)\,\kappa(dt) = \mathbf{Q}_m \int_{\mathbf{R}} h(t)1_{\{\hat{T}_b\circ\check{\theta}_t=0\}}f(Y_t)\,\kappa(dt).$$

If $\widehat{T}_b \circ \check{\theta}_t = 0$ and $f(Y_t) > 0$, then $d(Y_t, b) \geq \varepsilon$ and for every $\eta > 0,]t - \eta, t[\cap Z \neq \emptyset$. Consequently t is a discontinuity point of $s \to Y_s$. But Y is right continuous and E is a Lusin space and so Y has at most a countable number of discontinuities. Thus, since κ is diffuse, the above integrals vanish. Moreover $\int_{\mathbf{R}} \widehat{\mathbf{P}}^x[T_b = 0]f(x)\mu(dx)$ is precisely the first integral in the above display; see [G90, 8.21)]. Since μ is an arbitrary finite measure not charging m-semipolars, it follows from a theorem of Dellacherie [D88, p. 70] that $\{x \in \Gamma : \widehat{\mathbf{P}}^x[T_b = 0] > 0\}$ is m-semipolar. Letting $\varepsilon \downarrow 0$ through a sequence completes the proof of (A.6). $\square$

(A.7) Proposition. *Fix $B \in \mathcal{E}$ and define, for $\lambda \geq 0$ and $f \in p\mathcal{E}^*$,*

$$V^\lambda f = \mathbf{P}^\bullet \int_0^{T_B} e^{-\lambda t} f(X_t)dt \text{ and } \widehat{V}^\lambda f = \widehat{\mathbf{P}}^\bullet \int_0^{T_B} e^{-\lambda t} f(X_t)dt.$$

Then $(f, V^\lambda g) = (\widehat{V}^\lambda f, g)$.

Proof. It suffices to suppose f and g bounded. Then

$$\begin{aligned}
\mathbf{P}^m[f(X_0)g(X_t); T_B < t] &= \mathbf{P}^m\left[f(X_0)g(X_t); X_s \in B \text{ for some } s \in]0,t[\right] \\
&= \mathbf{Q}_m\left[f(Y_0)g(Y_t); Y_s \in B \text{ for some } s \in]0,t[\right] \\
&= \mathbf{Q}_m\left[f(Y_{-t})g(Y_0); Y_s \in B \text{ for some } s \in]-t,0[\right] \\
&= \widehat{\mathbf{P}}^m[g(\widehat{X}_0)f(\widehat{X}_t); \widehat{T}_B < t].
\end{aligned}$$

Multiply by $e^{-\lambda t}$ and integrate over $[0,\infty[$ to obtain

$$\int_E m(dx)f(x)\mathbf{P}^x \int_{T_B}^\infty e^{-\lambda t} g(X_t)dt = \int_E m(dx)g(x)\widehat{\mathbf{P}}^x \int_{T_B}^\infty e^{-\lambda t} f(X_t)dt.$$

Now $U^\lambda f = V^\lambda f + \mathbf{P}^\bullet \int_{T_B}^\infty e^{-\lambda t} f(X_t)dt$ with a similar formula for $\widehat{U}^\lambda f$ and since $(f, U^\lambda g) = (\widehat{U}^\lambda f, g)$ we obtain $(f, V^\lambda g) = (\widehat{V}^\lambda f, g)$. □

(A.8) Remark. Of course $U^\lambda f = V^\lambda f + \mathbf{P}_B^\lambda U^\lambda f$ since X has the strong Markov property. However since $\widehat{T}_B$ need not be predictable for $\widehat{X}$, it does not follow that $\widehat{\mathbf{P}}^\bullet \int_{T_B}^\infty e^{-\lambda t} f(X_t)dt = \widehat{\mathbf{P}}_B^\lambda \widehat{U}^\lambda f$ off an m-polar set.

We next prove (3.3).

(A.9) Proposition. *Using the notation of (3.3), $\widehat{U}_\ell^\lambda f = \widehat{\varphi}^\lambda \widehat{U}_\ell^\lambda f(b)$ if (i) $\lambda \geq 0$ and $f \in p\mathcal{E}^*$ or (ii) $\lambda > 0$ and $f \in bp\mathcal{E}$.*

Proof. It suffices to prove this for $\lambda > 0$ and $f \in bp\mathcal{E}^*$. Now $d\hat{\ell}_t$ is carried by $\{t : \widehat{X}_t = b\}$, so

$$\widehat{U}_\ell^\lambda f = \widehat{\mathbf{P}}^\bullet \int_{T_b}^\infty e^{-\lambda t} f(X_t) d\ell_t = \lim_{\varepsilon \downarrow 0} \widehat{\mathbf{P}}^\bullet \int_{T_b + \varepsilon} e^{-\lambda t} f(X_t) d\ell_t.$$

But for $\varepsilon > 0, \widehat{T}_b + \varepsilon$ is predictable for $\widehat{X}$ and so by the moderate Markov property, $\widehat{\mathbf{P}}^\bullet \int_{T_b+\varepsilon} e^{-\lambda t} f(X_t) d\ell_t = \widehat{\mathbf{P}}^\bullet [e^{-\lambda(T_b+\varepsilon)} \widehat{U}_\ell^\lambda f(X_{T_b+\varepsilon})]$. Since $\widehat{U}_\ell^\lambda f$ is λ-coexcessive (i.e. λ-excessive for $\widehat{X}$), $t \to \widehat{U}_\ell^\lambda f(\widehat{X}_t)$ has right limits on $[0, \infty[$ and left limits on $]0, \infty[$. See Lemma 2 in [CG79] and also the paragraph above (2.6) in [G99]. There exists a sequence (t_n) depending on $\widehat{\omega}$ decreasing to $\widehat{T}_b$ with $\widehat{X}(t_n) = b$ for each n, a.s. on $\{\widehat{T}_b < \infty\}$. Consequently $\lim_{\varepsilon \downarrow 0} \widehat{U}_\ell^\lambda f(\widehat{X}_{\hat{T}_b + \varepsilon}) = \widehat{U}_\ell^\lambda f(b)$ a.s. on $\{\widehat{T}_b < \infty\}$. In view of the above this establishes (A.9). □

(A.10) Remark. Of course, as explained near the beginning of this section, (A.9) is short for $\widehat{U}_\ell^\lambda f = \widehat{\varphi}^\lambda \widehat{U}_\ell^\lambda f(b)$ off an m–polar set and $\widehat{U}_\ell^\lambda f(b)$ is uniquely determined.

A similar argument yields the following:

(A.11) Corollary. *Let $\lambda > 0$ and $f \in pb\mathcal{E}$. Then $\widehat{U}^\lambda f = \widehat{V}^\lambda f + \widehat{U}^\lambda f(b) \widehat{\varphi}^\lambda$.*

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