On Joint Fourier-Laplace Transforms

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Abstract

Weak convergence of random variables is characterized by pointwise convergence of the Fourier transform of the respective distributions, and in some cases can also be characterized through the Laplace transform. For some distributions, the Laplace transform is easier to compute and provides an alternative approach to the method of characteristic functions that facilitates proving weak convergence. We show that for a bivariate distribution, a joint Fourier-Laplace transform always characterizes the distribution when the second variate is positive almost surely.

The joint Fourier-Laplace transform of bivariate random variables can sometimes be an analytically convenient metrization of weak convergence. Given the existence in a suitable interval of the Laplace transform of the second variate's distribution, which is up to sign the moment generating function of the variable, such a function characterizes the distribution and can be used to prove weak convergence of a sequence of variables. We focus on the joint Fourier-Laplace transform, which is essentially a characteristic function in the first variable and a moment generating function in the second. Below, we present the main analytical result that justifies the use of such transforms. We briefly mention some applications.

It sometimes occurs in statistical problems that the joint convergence of sample mean and sample variance must be determined. In this case we have a pair of random variables for which the second variate (when uncentered) is positive with probability one. Therefore the Laplace transform is typically well-defined for the second variate, and a joint Fourier-Laplace transform may be easier to calculate then a straight Fourier transform. See McElroy and Politis (2006a, 2006b) for some examples.

We now consider a pair of random variables (X, Y) such that $Y \geq 0$ almost surely. The joint Fourier-Laplace (FL) transform of X and Y is defined for $t \in \mathbb{R}$ and $u \in U = [0, \infty)$ by

$$\phi(t, u) := \mathbf{E}[\exp(itX - uY)].$$

In some cases, the moment generating function $u \mapsto \phi(0, u)$ is defined for $U = (-s_0, s_0)$ or $U = [0, s_0)$ for some $s_0 > 0$, but we will focus on the case $U = [0, \infty)$. The following theorem summarizes the important properties of the FL transform as concerns weak convergence.

Theorem 1 *(i) The joint distribution of (X, Y) is uniquely determined by the Fourier-Laplace transform ϕ.*

(ii) If (X_n, Y_n) is a sequence of random pairs with $Y_n \geq 0$ almost surely for each n, then (X_n, Y_n) converges weakly (to some pair (X, Y)) provided the sequence (ϕ_n) of Fourier-Laplace transforms converges pointwise on $\mathbb{R} \times [0, \infty)$ to a function ϕ on $\mathbb{R} \times [0, \infty)$ that is continuous at $(0, 0)$. In this case, $Y \geq 0$ almost surely, and the Fourier-Laplace transform of (X, Y) is ϕ.

Remark 1 By an example in Mukherjea, Rao, and Suen (2006), the converse of the second statement of Theorem 1 need not be true, i.e., there exist sequences of distributions whose characteristic functions converge, but whose moment generating functions do not.

Proof of Theorem 1. (i) For fixed real t and x, define

$$f(y, z) = e^{itx} e^{-zy}, \qquad y \geq 0, z \in W,$$

where $W = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0\}$. Then $f(y, \cdot)$ is analytic in the open set W, with derivative $\frac{\partial}{\partial z} f(y, z) = -y f(y, z)$. Notice that $|\frac{\partial}{\partial z} f(y, z)| = y e^{-\mathrm{Re}(z) y} \leq [e \mathrm{Re}(z)]^{-1}$, since $\mathrm{Re}(z) > 0$. Also, $f(\cdot, z)$ is measurable and the magnitude of f is bounded by 1. It follows from these observations and exercise 16.6 of Billingsley (1995) that $z \mapsto \mathbf{E}[\exp(tX - zY)]$ is analytic on W. Moreover, by the Dominated Convergence Theorem, the extension $\Phi(t, z) := \mathbf{E}[\exp(tX - zY)]$ of ϕ is continuous on $\mathbb{R} \times \overline{W}$, where $\overline{W} = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq 0\}$ is the closure of W.

Now suppose that (X^*, Y^*) is a second pair of random variables (such that $Y^* \geq 0$ almost surely) with FL transform equal to ϕ on all of $\mathbb{R} \times [0, \infty)$. Then for each real t, the two expectations $\mathbf{E}[\exp(itX - zY)]$ and $\mathbf{E}[\exp(itX^* - zY^*)]$ are analytic functions of $z \in W$ and coincide (with $\phi(t,\cdot)$) on the axis $\{z \in W \,|\, \mathrm{Im}(z) = 0\}$. It follows that

$$\mathbf{E}[\exp(itX - zY)] = \mathbf{E}[\exp(itX^* - zY^*)], \qquad (t,z) \in \mathbb{R} \times W.$$

By the continuity noted earlier this equality extends to $\mathbb{R} \times \overline{W}$. Thus, writing $z = -iv$ for complex z with $\mathrm{Re}(z) = 0$,

$$\mathbf{E}[\exp(itX + ivY)] = \mathbf{E}[\exp(itX^* + ivY^*)], \qquad \forall (t,v) \in \mathbb{R}^2.$$

Thus (X,Y) and (X^*,Y^*) have the same bivariate characteristic function, hence the same distribution.

(ii) The assumptions imply that the limits

$$\lim_n \mathbf{E}[\exp(itX_n)] = \lim_n \phi_n(t,0) = \phi(t,0), \qquad t \in \mathbb{R},$$

and

$$\lim_n \mathbf{E}[\exp(-uY)] = \lim_n \phi_n(0,u) = \phi(0,u), \qquad u \geq 0,$$

exist and are continuous at the origin. It follows that both (X_n) and (Y_n) converge in distribution. (See Theorem 2, page 431, in Feller (1971).) This in turn implies that both (X_n) and (Y_n) are tight. Consequently, the bivariate sequence (X_n, Y_n) is tight. Since $\phi(t,u) = \lim_n \phi(t,u)$ for all $(t,u) \in \mathbb{R} \times [0,\infty)$, all subsequential weak limits of (X_n, Y_n) have the same FL transform (namely ϕ), hence the same distribution, by assertion (i). It follows from the corollary on page 381 of Billingsley (1995) that (X_n, Y_n) converges in distribution to a limit (X,Y) with FL transform ϕ, and clearly $Y \geq 0$ almost surely. □

References

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