On General Perturbations of Symmetric Markov Processes

Z.-Q. Chen[*] **P. J. Fitzsimmons** **K. Kuwae**[†] and **T.-S. Zhang**[‡]

(October 20, 2008)

Abstract

Let X be a symmetric right process, and let $Z = \{Z_t, t \geq 0\}$ be a multiplicative functional of X that is the product of a Girsanov transform, a Girsanov transform under time-reversal and a continuous Feynman-Kac transform. In this paper we derive necessary and sufficient conditions for the strong L^2-continuity of the semigroup $\{T_t, t \geq 0\}$ given by $T_t f(x) = \mathbf{E}_x [Z_t f(X_t)]$, expressed in terms of the quadratic form obtained by perturbing the Dirichlet form of X in the appropriate way. The transformations induced by such Z include all those treated previously in the literature, such as Girsanov transforms, continuous and discontinuous Feynman-Kac transforms, and generalized Feynman-Kac transforms.

1 Introduction

We show that the main results of [6] concerning transformations of type (1.5) below can be improved by replacing the "Hardy class" conditions imposed there by weaker form-boundedness conditions. The key point, used already in much the same way in [4], is the observation that if μ is a smooth measure then there is a nest $\{F_k\}$ such that $\mathbf{1}_{F_k}\mu$ lies in the Kato class for each k.

To state things precisely we need to recall some facts and notation from [6]. Let $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, \mathbf{P}_x, x \in E)$ be an m-symmetric right Markov process on a Lusin space E, where m is a positive σ-finite measure with full topological support on E. A cemetery state ∂ is added to E to form $E_\partial := E \cup \{\partial\}$, and Ω is the totality of right-continuous, left-limited sample paths from $[0, \infty[$ to E_∂ that hold the value ∂ once attaining it. For any $\omega \in \Omega$, we set $X_t(\omega) := \omega(t)$. Let $\zeta(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = \partial\}$ be the life time of X. As usual, $\mathcal{F}_\infty$ and $\mathcal{F}_t$ are the minimal augmented σ-algebras obtained from $\mathcal{F}^0_\infty := \sigma\{X_s \mid 0 \leq s < \infty\}$ and $\mathcal{F}^0_t := \sigma\{X_s \mid 0 \leq s \leq t\}$ under $\{\mathbf{P}_x : x \in E\}$. For a Borel subset B of E, $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$ (the *exit time* of B) is an $(\mathcal{F}_t)$-stopping time.

---

[*] The research of this author is supported in part by NSF Grant DMS-0600206.

[†] The research of this author is partially supported by a Grant-in-Aid for Scientific Research (C) No. 19540220 from Japan Society for the Promotion of Science.

[‡] The research of this author is supported by the British EPSRC.

The transition semigroup $\{P_t : t \geq 0\}$ of X is defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t) : t < \zeta], \qquad t \geq 0.$$

Each P_t may be viewed as an operator on $L^2(m)$, and taken as a whole these operators form a strongly continuous semigroup of self-adjoint contractions. The Dirichlet form associated with X is the bilinear form

$$\mathcal{E}(u, v) := \lim_{t \downarrow 0} t^{-1}(u - P_t u, v)_m$$

defined on the space

$$\mathcal{F} := \left\{ u \in L^2(E; m) \,\middle|\, \sup_{t>0}\ t^{-1}(u - P_t u, u)_m < \infty \right\}.$$

Here we use the notation $(f, g)_m := \int_E f(x)g(x)\, m(dx)$.

As discussed in [6], there is no loss of generality in assuming that X is an m-symmetric Hunt process on a locally compact metric space E, whose associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular in $L^2(m)$, and that m is a positive Radon measure on E with full topological support. We fix a *core* $\mathcal{D}$ of $(\mathcal{E}, \mathcal{F})$; that is, $\mathcal{D}$ is a dense subspace of $(\mathcal{E}, \mathcal{F})$, which is also dense in $C_0(E)$ with respect to the uniform norm. We recall that an increasing sequence of closed sets $\{F_k\}$ is an $\mathcal{E}$-nest if $\cup_{n=1}^{\infty} \mathcal{F}_{F_k}$ is $\mathcal{E}_1^{1/2}$-dense in $\mathcal{F}$, where $\mathcal{E}_1(u, u) := \mathcal{E}(u, u) + (u, u)_m$ and $\mathcal{F}_{F_k} := \{u \in \mathcal{F} \mid u = 0\ m\text{-a.e. on } E \setminus F_k\}$. A set $N \subset E$ is *exceptional* if there is an $\mathcal{E}$-nest $\{F_k\}$ with $E \setminus N \subset \cup_k F_k$. A property $S(x)$ depending on $x \in E$ is said to hold for quasi-every x (q.e. in brief) provided it holds for all x outside an exceptional set. A positive continuous additive functional (PCAF in abbreviation) of X (call it A) determines a measure $\nu = \nu_A$ on the Borel subsets of E via the formula

$$\int_E f(x)\nu(dx) = \uparrow \lim_{t \to 0} t^{-1} \mathbf{E}_m \left[\int_0^t f(X_s)\, dA_s \right], \tag{1.1}$$

in which $f : E \to [0, \infty]$ is Borel measurable. The measure ν is necessarily *smooth*, in the sense that ν charges no exceptional set and there is an $\mathcal{E}$-nest $\{F_k\}$ of closed subsets of E such that $\nu(F_k) < \infty$ for each $n \in \mathbb{N}$. Conversely, given a smooth measure ν, there is a unique PCAF A^ν such that (1.1) holds with $A = A^\nu$. In the sequel we refer to this bijection between smooth measures and PCAFs as the *Revuz correspondence*, and to ν as the Revuz measure of A^ν. The Revuz measure ν of a PCAF A^ν is said to be of the Kato class provided

$$\lim_{t \to 0} \|\mathbf{E}_\cdot[A_t^\nu]\|_\infty = 0. \tag{1.2}$$

We write $\mathbf{K}(X)$ for the Kato class and define $\mathbf{K}_0(X) := \{\nu \in \mathbf{K}(X) \mid \nu(E) < \infty\}$. It is known (see [2, Theorem 2.4] or [9, Proposition 2.2] that if μ is a smooth measure then there is a nest $\{F_k\}$ such that $\mathbf{1}_{F_k}\mu \in \mathbf{K}_0(X)$ for all n.

Let $\mathcal{F}_e$ denote the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$. It is known that every $u \in \mathcal{F}_e$ admits a quasi-continuous version; this version, still denoted u, will be used throughout the paper. For every $u \in \mathcal{F}_e$, $u(X_t)$ admits *Fukushima's decomposition*, which asserts the existence and uniqueness of a martingale additive functional M^u of finite energy and a continuous additive function N^u of zero energy such that

$$u(X_t) - u(X_0) = M_t^u + N_t^u \qquad \text{for } t \in [0, \infty[, \tag{1.3}$$

$\mathbf{P}_x$-a.s. for $\mathcal{E}$-q.e. $x \in E$.

Let ζ_i denote the totally inaccessible part of the lifetime ζ, and define $I(\zeta) := [\![0, \zeta[\![\cup [\![\zeta_i]\!]$. If M is a locally square-integrable (local) martingale additive functional on $I(\zeta)$ (with respect to X), then the process $\langle M \rangle$ (the predictable quadratic variation of M) is a PCAF, and the associated Revuz measure (as in (1.1)) is denoted by $\mu_{\langle M \rangle}$. More generally, if M^u is the martingale part in the Fukushima decomposition (1.3) of $u \in \mathcal{F}$, then $\langle M^u, M \rangle$ is a CAF locally of bounded variation, and we have the associated Revuz measure $\mu_{\langle M^u, M \rangle}$, which is locally the difference of smooth (positive) measures.

Now let M and $\widehat{M}$ be two locally square-integrable local martingale additive functionals (MAFs) on $[\![0, \zeta[\![$, and let A be a CAF locally of bounded variation with (signed) Revuz measure μ. More precisely, $A = A^+ - A^-$, where A^+ and A^- are PCAFs with Revuz measure μ^+ and μ^-. We write $|\mu| := \mu^+ + \mu^-$ and (formally) $\mu = \mu^+ - \mu^-$. By the discussion of the preceding paragraph, there is a nest $\{F_k\}$ such that $\mathbf{1}_{F_k}(\mu_{\langle M \rangle} + \mu_{\langle \widehat{M} \rangle} + |\mu|) \in \mathbf{K}_0(X)$ for all n. The signed measure $\mu^+ - \mu^-$ is then well defined on each F_k.

Our main results concern the form perturbation $\mathcal{Q}$ of $(\mathcal{E}, \mathcal{F})$ defined on $\cup_k \mathcal{F}_{F_k}$ by

$$\begin{aligned} \mathcal{Q}(f,g) \;&=\; \mathcal{E}(f,g) - \int_E f(x)\, \mu_{\langle M^g, \widehat{M} \rangle}(dx) - \int_E g(x)\, \mu_{\langle M^f, M \rangle}(dx) - \int_E f(x) g(x)\, \mu(dx) \\ &\quad - \int_{E \times E} f(y) g(x) \varphi(x,y) \psi(y,x) N(x,dy) \mu_H(dx). \end{aligned} \tag{1.4}$$

Here φ and ψ are Borel functions defined on $E \times E$, vanishing on the diagonal and bounded below away from -1; these are the "jump functions" associated with M and $\widehat{M}$:

$$M_t - M_{t-} = \varphi(X_{t-}, X_t) \qquad \widehat{M}_t - \widehat{M}_{t-} = \psi(X_{t-}, X_t) \qquad \text{for every } t \in]0, \zeta[,$$

$\mathbf{P}_m$-a.e.; see [5, Corollary 2.9].

Let r_t denote the time-reversal operator defined on the path space Ω of X as follows: For $\omega \in \{t < \zeta\}$,

$$r_t(\omega)(s) = \begin{cases} \omega((t-s)-), & \text{if } 0 \le s < t, \\ \omega(0), & \text{if } s \ge t. \end{cases}$$

(It should be borne in mind that the restriction of the measure $\mathbf{P}_m$ to $\mathcal{F}_t \cap \{t < \zeta\}$ is invariant under r_t on $\Omega \cap \{\zeta > t\}$.) Now define, for $0 \le t < \zeta$,

$$Z_t = \operatorname{Exp}(M_t + A_t^\mu + \langle M^c, \widehat{M^c} \rangle_t) \cdot \operatorname{Exp}(\widehat{M}_t) \circ r_t \cdot (1 + \psi(X_t, X_{t-})), \tag{1.5}$$

wherein Exp denotes the familiar Doléans-Dade stochastic exponential: If Y is a semimartingale with $Y_0 = 0$, then $Z = \operatorname{Exp}(Y)$ is the unique solution of the SDE

$$Z_t = 1 + \int_0^t Z_{s-}\, dY_s,$$

and is given explicitly by the formula

$$\mathrm{Exp}(Y_t) = \exp\left(Y_t - \frac{1}{2}\langle Y^c, Y^c\rangle_t\right) \prod_{s\in]0,t]} (1+\Delta Y_s)e^{-\Delta Y_s}, \qquad t \geq 0.$$

Finally, define

$$T_t f(x) := \mathbf{E}_x\left[Z_t f(X_t)\right], \tag{1.6}$$

The following result is obtained in [6, Theorem 3.1]. For its statement recall that a signed smooth measure ν is said to be of the Hardy class $\mathbf{H}(X)$ of X if there are constants $\delta > 0$ and $\gamma \geq 0$ such that

$$\int_E u(x)^2\, |\nu|(dx) \leq \delta \cdot \mathcal{E}(u,u) + \gamma \cdot (u,u)_m \qquad \text{for every } u \in \mathcal{F},$$

where $|\nu| := \nu^+ + \nu^-$ denotes the total variation measure of ν. Note that every $\nu \in \mathbf{H}(X)$ is a Radon measure. It is known that $\mathbf{K}(X) \subset \mathbf{H}(X)$. Also, we write $\mathcal{F}_b$ for the class of (m-essentially) bounded elements of $\mathcal{F}$.

Theorem 1.1 *Assume that* $\mu_{\langle M\rangle}, \mu_{\langle\widehat{M}\rangle}$ *and* $|\mu|$ *are all in the Hardy class* $\mathbf{H}(X)$*, and that there are constants* $\alpha > 0$ *and* $c > 1$ *such that*

$$c^{-1}\mathcal{E}_1(u,u) \leq \mathcal{Q}_\alpha(u,u) \leq c\,\mathcal{E}_1(u,u) \qquad \text{for } u \in \mathcal{F}_b.$$

Then $\{T_t, t \geq 0\}$ *defined by (1.6) coincides with the strongly continuous semigroup associated with* $(\mathcal{Q}, \mathcal{F})$ *in* $L^2(E;m)$.

As a consequence of the above result, the following result is established in [6] as Theorem 5.1.

Theorem 1.2 *Suppose that* M *is a locally square integrable martingale additive functional of* X *with anti-symmetric jump function* φ *such that* $\mu_{\langle M\rangle} \in \mathbf{H}(X)$*. Define*

$$\mathcal{Q}(f,g) := \mathcal{E}(f,g) + \frac{1}{2}\mu_{\langle M^{fg}, M\rangle}(E) \qquad \text{for } f,g \in \mathcal{F}_b.$$

Suppose there are constants $\alpha > 0$ *and* $c > 1$ *so that*

$$c^{-1}\mathcal{E}_1(u,u) \leq \mathcal{Q}_\alpha(u,u) \leq c\,\mathcal{E}_1(u,u) \qquad \text{for } u \in \mathcal{F}_b.$$

Then $\overline{P}_t f(x) := \mathbf{E}_x[e^{\Lambda(M)_t} f(X_t)]$ *is the symmetric semigroup associated with* $(\mathcal{Q}, \mathcal{F})$*, where*

$$\Lambda(M)_t := -\frac{1}{2}(M_t + M_t \circ r_t + \varphi(X_t, X_{t-})), \qquad 0 < t < \zeta.$$

For $u \in \mathcal{F}_e$, M^u is a MAF of X having finite energy with anti-symmetric jump function $u(x) - u(y)$ and $\Lambda(M^u) = N^u$. The following corollary is an immediate consequence of Theorem 1.2.

Corollary 1.3 *Suppose $u \in \mathcal{F}_e$ with $\mu_{\langle M^u \rangle} \in \mathbf{H}(X)$. Define*

$$\mathcal{Q}(f,g) := \mathcal{E}(f,g) + \mathcal{E}(fg,u) \qquad \text{for } f,g \in \mathcal{F}_b.$$

Suppose there are constants $\alpha > 0$ and $c > 1$ so that

$$c^{-1}\mathcal{E}_1(u,u) \leq \mathcal{Q}_\alpha(u,u) \leq c\,\mathcal{E}_1(u,u) \qquad \text{for } u \in \mathcal{F}_b.$$

Then $\overline{P}_t f(x) := \mathbf{E}_x[e^{N_t^u} f(X_t)]$ is the strongly continuous symmetric semigroup in $L^2(E;m)$ associated with $(\mathcal{Q},\mathcal{F})$.

The purpose of this paper is to establish the following extension of Theorem 1.1.

Theorem 1.4 *Let $\mu_{\langle M \rangle}$, $\mu_{\langle \widehat{M} \rangle}$, and $|\mu|$ be as in the preceding discussion, and let $\{F_k\}$ be an $\mathcal{E}$-nest such that $\mathbf{1}_{F_k}\left(\mu_{\langle M \rangle} + \mu_{\langle \widehat{M} \rangle} + |\mu|\right)$ is in the Kato class, for each $k \geq 1$. Suppose that the quadratic form defined on $\cup_k \mathcal{F}_{F_k}$ by (1.4) is bounded below in the sense that there is a constant $\alpha_0 \geq 0$ such that for every $f \in \cup_k \mathcal{F}_{F_k}$*

$$\mathcal{Q}_{\alpha_0}(f,f) \geq 0. \tag{1.7}$$

Suppose also that there is a constant K such that for every $f,g \in \cup_k \mathcal{F}_{F_k}$

$$|\mathcal{Q}_{\alpha_0}(f,g)| \leq K\,\mathcal{Q}_{\alpha_0}(f,f)^{1/2}\mathcal{Q}_{\alpha_0}(g,g)^{1/2}. \tag{1.8}$$

The formula (1.6) then defines a strongly continuous semigroup $\{T_t : t \geq 0\}$ of bounded operators on $L^2(m)$. Let $(\mathcal{C},\mathcal{D}(\mathcal{C}))$ be a sectorial closed form bounded below by $-\alpha_0$ in the sense that $\mathcal{C}_{\alpha_0}(f,f) \geq 0$ and $\mathcal{C}_{\alpha_0}(f,g) \leq L\mathcal{C}_{\alpha_0}(f,f)^{1/2}\mathcal{C}_{\alpha_0}(g,g)^{1/2}$ for all $f,g \in \mathcal{D}(\mathcal{C})$, for some $L > 0$. If $\{T_t, t \geq 0\}$ is associated with $(\mathcal{C},\mathcal{D}(\mathcal{C}))$, then $(\mathcal{C},\mathcal{D}(\mathcal{C}))$ is the largest closed form bounded below by $-\alpha_0$ that is less than $(\mathcal{Q}, \bigcup_{k\geq 1}\mathcal{F}_{F_k})$ on the diagonal:

(i) $\bigcup_k \mathcal{F}_{F_k} \subset \mathcal{D}(\mathcal{C})$ and $\mathcal{Q}(f,f) \geq \mathcal{C}(f,f)$ for every $f \in \cup_{k\geq 1}\mathcal{F}_{F_k}$.

(ii) If $(\mathcal{S},\mathcal{D}(\mathcal{S}))$ is a closed sectorial form bounded below and enjoying property in (i) with $(\mathcal{S},\mathcal{D}(\mathcal{S}))$ in place of $(\mathcal{C},\mathcal{D}(\mathcal{C}))$, then $\mathcal{D}(\mathcal{C}) \subset \mathcal{D}(\mathcal{S})$ and $\mathcal{C}(f,f) \geq \mathcal{S}(f,f)$ for every $f \in \mathcal{D}(\mathcal{C})$.

Conversely, if $\{T_t, t \geq 0\}$ defined by (1.6) is a strongly continuous semigroup on $L^2(m)$ with $\|T_t\| \leq e^{\alpha_0 t}$ for some $\alpha_0 \geq 0$, then (1.7) holds, provided $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest such that

$$\mathbf{1}_{F_k}\left(\mu_{\langle M \rangle} + \mu_{\langle \widehat{M} \rangle} + |\mu|\right) \in \mathbf{K}_0(X) \qquad \text{for every } k \geq 1.$$

A smooth measure ν is said to be of the *local Kato class* $\mathbf{K}_{loc}(X)$ of X if for each compact set K, $\mathbf{1}_K \nu \in \mathbf{K}(X)$. Since every measure in $\mathbf{K}(X)$ is a Radon measure, $\nu \in \mathbf{K}_{loc}(X)$ if and only if $\mathbf{1}_K \nu \in \mathbf{K}_0(X)$ for every compact K.

Corollary 1.5 *Suppose $\mu_{\langle M \rangle} + \mu_{\langle \widehat{M} \rangle} + |\mu| \in \mathbf{K}_{loc}(X)$. Assume that there exists $\alpha_0 \geq 0$ such that (1.7) and (1.8) hold for every $f,g \in \mathcal{D}$. Then the same conclusion holds as in Theorem 1.4.*

Remark 1.6 (i) Note that under condition (1.7), the sector condition (1.8) is automatically satisfied when $\mathcal{Q}$ is symmetric.

(ii) For $n \geq 1$, define $T_t^{(n)} f(x) = E_x[Z_t f(X_t); t < \tau_{F_n}]$. Let $G_\alpha^{(n)}$ be the associated resolvent. The condition (1.8) yields the following: For each $n \in \mathbb{N}$, there exists a dense subspace $\mathcal{D}_n$ of $L^2(F_n; m)$ such that

$$G_\alpha^{(n)}(\mathcal{D}_n) \subset L^2(F_n; m), \quad \text{for } \alpha > \alpha_0. \tag{1.9}$$

In Theorem 1.4 or Corollary 1.5, we can replace (1.8) with (1.9) to obtain the same conclusion.

(iii) For any strongly L^2-continuous semigroup $\{T_t, t \geq 0\}$ that is symmetric, there is some $\alpha_0 \geq 0$ such that $\|T_t\| \leq e^{\alpha_0 t}$. This can be proved as follows. Let $\alpha_0 = 0 \vee \log \|T_1\|$ and define $\widehat{T}_t = e^{-\alpha_0 t} T_t$. Then $\{\widehat{T}_t, t \geq 0\}$ is a strongly continuous symmetric semigroup with $\|\widehat{T}_1\| \leq 1$. Thus for every $f \in L^2$ with $\|f\|_2 = 1$,

$$\|\widehat{T}_{1/2} f\|_2^2 = (f, \widehat{T}_1 f) \leq 1.$$

This proves $\|\widehat{T}_{1/2}\| \leq 1$. Iterating, we have $\|\widehat{T}_{1/2^n}\| \leq 1$ for every integer $n \geq 1$. Using the semigroup property, we obtain $\|\widehat{T}_{k/2^n}\| \leq 1$ for integers $k, n \geq 1$. The strong continuity of $\{\widehat{T}_t, t \geq 0\}$, permits us to conclude that $\|\widehat{T}_t\| \leq 1$ and so $\|T_t\| \leq e^{-\alpha_0 t}$ for every $t \geq 0$. □

Section 2 contains the proof of Theorem 1.4, along with the following strengthened version of [6, Theorem 5.1]. For its statement let M be a locally square-integrable MAF on $[\![0, \zeta[\![$ with jump function φ such that

$$\int_0^t \int_E \varphi(y, X_s)^2 N(X_s, dy) dH_s < \infty \quad \text{for every } t < \zeta, \qquad \mathbf{P}_x\text{-a.s.}, \tag{1.10}$$

for $\mathcal{E}$-q.e. $x \in E$. The above condition is automatically satisfied when φ is anti-symmetric, as was assumed in Theorem 5.1 of [6]. Under the above condition, we have

$$\int_0^t \int_{E_\partial} \left(\varphi(X_s, y)^2 + \varphi(y, X_s)^2 \right) N(X_s, dy) dH_s < \infty \quad \text{for every } t < \zeta, \qquad \mathbf{P}_x\text{-a.s.},$$

for $\mathcal{E}$-q.e. $x \in E$. Thus by Lemma 3.2 of [5], there is a unique purely discontinuous local MAF K on $[\![0, \zeta[\![$ with jump function $-\varphi(x, y) - \varphi(y, x)$. Define $\Lambda(M)_0 = 0$ and

$$\Lambda(M)_t := -\frac{1}{2}(M_t + M_t \circ r_t + \varphi(X_t, X_{t-}) + K_t), \qquad 0 < t < \zeta.$$

Although it is not obvious, this formula defines a CAF of X; this crucial result is proved in [5, §2–3]. The map $M \mapsto \Lambda(M)$ extends an earlier construction of S. Nakao [12], and is the key to Nakao's stochastic integral (and its extension in [5]). Note that $\Lambda(M^u) = N^u$ on $[\![0, \zeta[\![$ for $u \in \mathcal{F}_e$.

The following quadratic form $\mathcal{Q}$ on $\mathcal{F}_b$ is a well-defined symmetric form provided $\mu_{\langle M\rangle} \in \mathbf{H}(X)$: for $f, g \in \mathcal{F}_b$,

$$\mathcal{Q}(f,g) = \mathcal{E}(f,g) + \frac{1}{2}\mu_{\langle M^{fg}, M\rangle}(E). \tag{1.11}$$

Define $\check{\varphi}(x,y) := \varphi(y,x)$ and $N(\mathbf{1}_{E\times E}\check{\varphi}^2)(x) := \int_E \check{\varphi}(x,y)^2 N(x,dy)$, and write $\mathcal{F}_{F_k,b}$ for the set of bounded elements of $\mathcal{F}_{F_k}$.

Theorem 1.7 *Let M be a locally square-integrable MAF on $[\![0,\zeta[\![$ with jump function φ satisfying condition (1.10). Let $\{F_k\}$ be an $\mathcal{E}$-nest such that*

$$\mathbf{1}_{F_k}\left(\mu_{\langle M\rangle} + N(\mathbf{1}_{E\times E}\check{\varphi}^2)\mu_H\right) \in \mathbf{K}_0(X) \qquad \text{for every } k \geq 1.$$

Then $\mathcal{Q}$ defined by (1.11) for $f, g \in \bigcup_{k\geq 1}\mathcal{F}_{F_k,b}$ is well-defined. Suppose that $(\mathcal{Q}, \bigcup_{k\geq 1}\mathcal{F}_{F_k,b})$ is bounded below in the sense that there is a constant $\alpha_0 \geq 0$ such that for every $f \in \bigcup_k \mathcal{F}_{F_k,b}$

$$\mathcal{Q}(f,f) + \alpha_0 (f,f)_{L^2} \geq 0. \tag{1.12}$$

Then the symmetric semigroup $\overline{P}_t f(x) := \mathbf{E}_x\left[e^{\Lambda(M)_t} f(X_t)\right]$ is strongly continuous on $L^2(m)$, and the associated quadratic form $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ is the largest closed symmetric form bounded below by $-\alpha_0$ that is less than $(\mathcal{Q}, \bigcup_{k\geq 1}\mathcal{F}_{F_k,b})$. Conversely, if $\overline{P}_t f(x) := \mathbf{E}_x\left[e^{\Lambda(M)_t} f(X_t)\right]$ is a strongly continuous semigroup on $L^2(E;m)$, then there is some $\alpha_0 \geq 0$ so that (1.12) holds.

Corollary 1.8 *Suppose $\mu_{\langle M\rangle} + N(\mathbf{1}_{E\times E}\check{\varphi}^2)\mu_H \in \mathbf{K}_{loc}(X)$. Assume that there exists a constant $\alpha_0 \geq 0$ such that (1.12) holds for every $f \in \mathcal{D}$. Then the same conclusion holds as in Theorem 1.7.*

Applying the above theorem to the special case of $M = M^u$ for $u \in \mathcal{F}_e$, we immediately have the following result, which was obtained in [8, Theorem 4.6] by a different method.

Corollary 1.9 *Let $u \in \mathcal{F}_e$. Suppose that the form $\mathcal{Q}$ defined by*

$$\mathcal{Q}(f,g) := \mathcal{E}(f,g) + \mathcal{E}(fg,u) \qquad \text{for } f, g \in \mathcal{F}_b,$$

is bounded below on $\mathcal{F}_b$ in the sense that there is a constant $\alpha_0 \geq 0$ such that

$$\mathcal{Q}_{\alpha_0}(f,f) \geq 0 \qquad \text{for every } f \in \mathcal{F}_b. \tag{1.13}$$

Then the symmetric semigroup $\overline{P}_t f(x) := \mathbf{E}_x\left[e^{N_t^u} f(X_t)\right]$ is strongly continuous on $L^2(m)$, and the associated quadratic form $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ is the largest closed symmetric form bounded below by $-\alpha_0$ that is less than $(\mathcal{Q}, \mathcal{F}_b)$. Conversely, if $\overline{P}_t f(x) := \mathbf{E}_x\left[e^{N_t^u} f(X_t)\right]$ is a strongly continuous semigroup on $L^2(E;m)$, then there is some $\alpha_0 \geq 0$ so that (1.13) holds.

Acknowledgement. We thank Z.-M. Ma and W. Sun for helpful comments and discussions.

2 Proofs

We prepare for the proof of Theorem 1.4 with a lemma. Define, for $f \in L^2_+(m)$, $G_\alpha f(x) := \mathbf{E}_x\left[\int_0^\infty e^{-\alpha t} Z_t f(X_t)\,dt\right]$. If G_α maps $L^2_+(m)$ into $L^2(m)$, then we extend G_α linearly to all of $L^2(m)$ in the obvious way. In this case we can define

$$\mathcal{D}(\mathcal{C}) := \left\{ u \in L^2(E;m) : \sup_{\beta > 0} \beta(u - \beta G_{\beta+\alpha_0} u, u) < \infty \right\}. \tag{2.1}$$

Moreover, from [3], we can deduce that $u \in \mathcal{D}(\mathcal{C})$ if and only if $\sup_{t>0} \frac{1}{t}(u - e^{-\alpha_0 t} T_t u, u) < \infty$ provided $\{G_\alpha; \alpha > \alpha_0\}$ is associated to a sectorial closed form bounded below by $-\alpha_0$ on $L^2(E;m)$.

Lemma 2.1 *Assume that $\|\beta G_{\beta+\alpha_0}\| \le 1$ for some $\alpha_0 \ge 0$ and each G_α maps $L^2_+(m)$ into $L^2(m)$. If $u \in \mathcal{D}(\mathcal{C})$ then $\alpha G_\alpha u$ converges in $L^2(m)$ to u, as $\alpha \to \infty$.*

Proof. Let $u \in \mathcal{D}(\mathcal{C})$. In view of $\|\beta G_{\beta+\alpha_0}\| \le 1$, we have

$$\begin{aligned} &(u - \beta G_{\beta+\alpha_0} u, u - \beta G_{\beta+\alpha_0} u) \\ &= 2(u - \beta G_{\beta+\alpha_0} u, u) - [(u,u) - \|\beta G_{\beta+\alpha_0} u\|^2] \\ &\le 2(u - \beta G_{\beta+\alpha_0} u, u) \end{aligned}$$

By the definition of $\mathcal{D}(\mathcal{C})$, it follows that

$$\begin{aligned} &\overline{\lim_{\beta\to\infty}}\,(u - \beta G_{\beta+\alpha_0} u, u - \beta G_{\beta+\alpha_0} u) \\ &\le 2\,\overline{\lim_{\beta\to\infty}}\,(u - \beta G_{\beta+\alpha_0} u, u) \le 0, \end{aligned}$$

which proves the Lemma.

□

Proof of Theorem 1.4.

The proof is much the same as that of Theorem 3.1 in [6]. In steps (1)–(3) of that proof it was shown that $T_t^{(n)} f(x) := \mathbf{E}_x[Z_t \cdot f(X_t); t < \tau_{F_n}]$ is the semigroup associated with the form $(\mathcal{F}_{F_n}, \mathcal{Q})$. Because of (1.7), (1.8) (hence (1.9)) and [10, Proposition 3.1], the resolvent $G_\alpha^{(n)}$ associated with $T_t^{(n)}$ satisfies $\|G_\alpha^{(n)}\|_2 \le 1/(\alpha - \alpha_0)$ for all $\alpha > \alpha_0$. By the monotone convergence theorem, if $f \in L^2_+(m)$, then $G_\alpha^{(n)} f(x)$ increases to

$$G_\alpha f(x) := \mathbf{E}_x\left[\int_0^\infty e^{-\alpha t} Z_t f(X_t) dt\right]$$

as $n \to \infty$ for q.e. $x \in E$. This implies that $G_\alpha f$ is well defined for $f \in L^2_+(E;m)$ and that $\|G_\alpha f\|_2 \le \frac{1}{\alpha - \alpha_0}\|f\|_2$ for all $\alpha > \alpha_0$. Moreover, by the dominated convergence theorem, $G_\alpha^{(n)} f$ converges to $G_\alpha f$ in $L^2(m)$ first for every $f \in L^2_+(E;m)$ and then for every $f \in L^2(m)$.

Next we show that $\{G_\alpha, \alpha > \alpha_0\}$ is a strongly continuous resolvent on $L^2(m)$. Recall the definition (2.1). As $G_\alpha^{(k)} f \leq G_\alpha f$ for $f \geq 0$, we conclude that $\mathcal{F}_{F_k}^+ \subset \mathcal{D}(\mathcal{C})$ and hence $\bigcup_{k\geq 1} \mathcal{F}_{F_k} \subset \mathcal{D}(\mathcal{C})$. Because $\|\alpha G_\alpha\|_2 \leq \frac{\alpha}{\alpha-\alpha_0}$, the asserted strong continuity follows from Lemma 2.1. We now assume that $\{T_t, t \geq 0\}$ (or $\{G_\alpha, \alpha > \alpha_0\}$) is associated to a closed sectorial form $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ bounded below by $-\alpha_0$; the domain $\mathcal{D}(\mathcal{C})$ of this form is given by (2.1). We claim that $\mathcal{Q}(f,f) \geq \mathcal{C}(f,f)$ provided $f \in \mathcal{F}_{F_k}$. This is because for such f,

$$\begin{aligned} \mathcal{C}_{\alpha_0}(f,f) &= \sup_{t>0} t^{-1}(f - e^{-\alpha_0 t} T_t f, f) = \sup_{t>0} \lim_{n\to\infty} t^{-1}(f - e^{-\alpha_0 t} T_t^{(n)} f, f) \\ &\leq \lim_{n\to\infty} \sup_{t>0} t^{-1}(f - e^{-\alpha_0 t} T_t^{(n)} f, f) = \lim_{n\to\infty} \mathcal{Q}_{\alpha_0}(f,f) = \mathcal{Q}_{\alpha_0}(f,f). \end{aligned}$$

Now suppose $(\mathcal{S}, \mathcal{D}(\mathcal{S}))$ is another closed sectorial form bounded below by $-\alpha_0$ that is less than $(\mathcal{Q}, \cup_{k\geq 1}\mathcal{F}_{F_k})$ in the sense of point (ii) of Theorem 1.4. Fix $\alpha > \alpha_0$, $k \in \mathbb{N}$, $u \in \mathcal{F}_{F_k}$, and set $f := G_\alpha u$. Then we know that $f_n := G_\alpha^{(n)} u \in \mathcal{F}_{F_n}$ converges in $L^2(m)$ to f as $n \to \infty$. It follows that

$$\sup_{n\geq 1} \mathcal{S}_\alpha(f_n, f_n) \leq \sup_{n\geq 1} \mathcal{Q}_\alpha(f_n, f_n) = \sup_{n\geq 1}(u, G_\alpha^{(n)} u) < \infty.$$

This implies that $f \in \mathcal{D}(\mathcal{S})$ and that

$$\begin{aligned} \mathcal{S}_\alpha(f,f) &\leq \varliminf_{n\to\infty} \mathcal{S}_\alpha(f_n, f_n) \leq \varliminf_{n\to\infty} \mathcal{Q}_\alpha(f_n, f_n) = \lim_{n\to\infty}(u, G_\alpha^{(n)} u) \\ &= (u, G_\alpha u) = \mathcal{C}_\alpha(G_\alpha u, G_\alpha u) = \mathcal{C}_\alpha(f,f). \end{aligned}$$

It follows that $\mathcal{D}(\mathcal{C}) \subset \mathcal{D}(\mathcal{S})$ and $\mathcal{S}(f,f) \leq \mathcal{C}(f,f)$. This proves that $(\mathcal{S}, \mathcal{D}(\mathcal{S}))$ less than $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$. Thus when $\{T_t, t \geq 0\}$ is associated with a closed sectorial form $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ bounded below, $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ is *the* largest closed sectorial form bounded below that is less than $(\mathcal{Q}, \cup_{k\geq 1}\mathcal{F}_{F_k})$ on diagonal. This proves the first part of Theorem 1.4.

Turning to the second part of Theorem 1.4, suppose that $\{T_t : t \geq 0\}$ defined by (1.6) is a strongly continuous semigroup on $L^2(m)$ with $\|T_t\| \leq e^{\alpha_0 t}$. Let $\{F_k\}$ be an $\mathcal{E}$-nest such that $\mathbf{1}_{F_k}\left(\mu_{\langle M\rangle} + \mu_{\langle \widehat{M}\rangle} + |\mu|\right)$ is in the Kato class for every $k \geq 1$. Note that for every $n \geq 1$, the semigroup $\{T_t^{(n)}, t \geq 0\}$ defined by

$$T_t^{(n)} f := \mathbf{E}_x\left[Z_t f(X_t); t < \tau_{F_n}\right]$$

is the strongly continuous semigroup associated with the closed quadratic form $(\mathcal{Q}, \mathcal{F}_{F_n})$. For each $f \in L^2(F_n; m)$ and each $t \geq 0$,

$$\|T_t^{(n)} f\|_2 \leq \|T_t^{(n)}|f|\|_2 \leq \|T_t|f|\|_2 \leq e^{\alpha_0 t}\|f\|_2.$$

Thus $\|T_t^{(n)}\|_2 \leq e^{\alpha_0 t}$, and consequently the associated resolvent $\{G_\alpha^{(n)}, \alpha > \alpha_0\}$ satisfies the estimate

$$\|G_\alpha^{(n)}\|_2 \leq \frac{1}{\alpha - \alpha_0} \qquad \text{for } \alpha > \alpha_0.$$

It follows then for $f \in L^2(F_n; m)$,

$$\alpha(f - \alpha G_\alpha f, f)_{L^2(F_n;m)} + \alpha_0(\alpha G_\alpha^{(n)} f, f)_{L^2(F_n;m)} \geq 0.$$

Consequently,

$$\mathcal{Q}_{\alpha_0}(f,f)\geq 0 \qquad \text{for every } f\in\mathcal{F}_{F_n}.$$

This proves that (1.7) holds. □

Proof of Theorem 1.7.

Let M be a locally square integrable MAF of X on $[\![0,\zeta[\![$ with jump function φ satisfying condition (1.10). Observe that, unlike the case considered in Theorem 5.1 of [6], we do not assume φ is anti-symmetric.

Note that $M=M^c+M^d$, where M^c and M^d are the continuous and purely discontinuous martingale parts of M. Let $M^{d,1}$ and $M^{d,2}$ be the purely discontinuous locally square integrable MAFs of X on $[\![0,\zeta[\![$ with

$$M_t^{d,1}-M_{t-}^{d,1}=\mathbf{1}_{\{|\varphi(X_{t-},X_t)|\leq 1\}}\varphi(X_{t-},X_t)$$

and

$$M_t^{d,2}-M_{t-}^{d,2}=\mathbf{1}_{\{|\varphi(X_{t-},X_t)|>1\}}\varphi(X_{t-},X_t),$$

respectively. Let K^1 and K^2 be the purely discontinuous locally square integrable MAFs of X on $[\![0,\zeta[\![$ with

$$K_t^1-K_{t-}^1=-\mathbf{1}_{\{|\varphi(X_{t-},X_t)|\leq 1\}}\varphi(X_{t-},X_t)-\mathbf{1}_{\{|\varphi(X_t,X_{t-})|\leq 1\}}\varphi(X_t,X_{t-})$$

and

$$K_t^2-K_{t-}^2=-\mathbf{1}_{\{|\varphi(X_{t-},X_t)|>1\}}\varphi(X_{t-},X_t)-\mathbf{1}_{\{|\varphi(X_t,X_{t-})|>1\}}\varphi(X_t,X_{t-}).$$

Note that $K:=K^1+K^2$ is the purely discontinuous locally square integrable MAF of X on $[\![0,\zeta[\![$ with jump function $-\varphi(x,y)-\varphi(y,x)$.

Let (N,H) be a Lévy system of X. By (2.8) in [6], for $t<\zeta$,

$$\begin{aligned}\Lambda(M^{d,2})_t &= -\frac{1}{2}\left(M_t^{d,2}+M_t^{d,2}\circ r_t+\varphi(X_t,X_{t-})\mathbf{1}_{\{|\varphi(X_t,X_{t-})|>1\}}+K_t^2\right)\\ &= \frac{1}{2}\int_0^t\int_E \mathbf{1}_{\{|\varphi(X_s,y)|>1\}}\varphi(X_s,y)N(X_s,dy)\,dH_s\\ &\quad -\frac{1}{2}\int_0^t\int_E \mathbf{1}_{\{|\varphi(y,X_s)|>1\}}\varphi(y,X_s)N(X_s,dy)\,dH_s.\end{aligned}$$

Thus $\Lambda(M^{d,2})$ is CAF of X with signed Revuz measure

$$\nu(dx):=\frac{1}{2}\int_{E\times E}\left(\varphi(x,y)\mathbf{1}_{\{|\varphi(x,y)|>1\}}-\varphi(y,x)\mathbf{1}_{\{|\varphi(y,x)|>1\}}\right)N(x,dy)\mu_H(dx).$$

Note that

$$|\nu|(dx)\leq\left(\frac{1}{2}\int_E\varphi(x,y)^2N(x,dy)+\frac{1}{2}\int_E\varphi(y,x)^2N(x,dy)\right)\mu_H(dx).$$

Put $M^1 := M^c + M^{d,1}$, a locally square integrable MAF of X with bounded jumps. Then

$$\Lambda(M) = \Lambda(M^1) + \Lambda(M^{d,2}) = \Lambda(M^1) + A^{\nu}.$$

Let $J_t^{(1)}$ be the purely discontinuous locally square-integrable MAF on $[\![0,\zeta[\![$ satisfying

$$\Delta J_t^{(1)} = \exp\left(-\frac{1}{2}\varphi(X_{t-}, X_t)\mathbf{1}_{\{|\varphi(X_{t-},X_t)|\le 1\}}\right) - 1, \qquad t \in]0,\zeta[.$$

Then $J_t^{(1)} + \frac{1}{2}M^{d,1}$ is the purely discontinuous locally square-integrable MAF on $[\![0,\zeta[\![$ with jump function

$$\psi(x,y) := \exp\left(-\frac{1}{2}\varphi(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le 1\}}\right) - 1 + \frac{1}{2}\varphi(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le 1\}}.$$

Note that by Taylor expansion,

$$|\psi(x,y)| \le c|\varphi(x,y)|^2\mathbf{1}_{\{|\varphi(x,y)|\le 1\}}$$

and so $\int_0^t\int_E |\psi(X_s,y)|N(X_s,dy)dH_s < \infty$ $\mathbf{P}_x$-a.s. on $[\![0,\zeta[\![$ for $\mathcal{E}$-q.e. $x \in E$. Therefore,

$$J_t^{(1)} + \frac{1}{2}M^{d,1} = \sum_{s\le t}\psi(X_{s-},X_s) - \int_0^t\int_E \psi(X_s,y)N(X_s,dy)dH_s \qquad \text{for } t \in [0,\zeta[$$

$\mathbf{P}_x$-a.s. for $\mathcal{E}$-q.e. $x \in E$. Set $\overline{M}_t := J_t^{(1)} - \frac{1}{2}M_t^c$ for $t \in [0,\zeta[$. We then have, on $\{t < \zeta\}$,

$$e^{-\frac{1}{2}M_t^1} = e^{\overline{M}_t - (J_t^{(1)} + \frac{1}{2}M_t^{d,1})} = \mathrm{Exp}(\overline{M})e^{A_t^{\mu_1}},$$

where

$$\mu_1(dx) := \int_E \psi(x,y)N(x,dy)\mu_H(dx) + \frac{1}{8}\mu_{\langle M^c\rangle}(dx).$$

Let $J_t^{(2)}$ be the purely discontinuous locally square-integrable MAF on $[\![0,\zeta[\![$ with jump function

$$\exp\left(\frac{1}{2}\varphi(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le 1\}} + \frac{1}{2}\check{\varphi}(x,y)\mathbf{1}_{\{|\check{\varphi}(x,y)|\le 1\}}\right) - 1$$

Then $J_t^{(2)} + \frac{1}{2}K_t^1$ is the purely discontinuous locally square-integrable MAF on $[\![0,\zeta[\![$ with jump function

$$\begin{aligned}\psi^{(2)}(x,y) \quad := \quad & \exp\left(\frac{1}{2}\varphi(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le 1\}} + \frac{1}{2}\check{\varphi}(x,y)\mathbf{1}_{\{|\check{\varphi}(x,y)|\le 1\}}\right) - 1 \\ & - \left(\frac{1}{2}\varphi(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le 1\}} + \frac{1}{2}\check{\varphi}(x,y)\mathbf{1}_{\{|\check{\varphi}(x,y)|\le 1\}}\right).\end{aligned}$$

Note that by Taylor expansion,

$$|\psi^{(2)}(x,y)| \le c|\varphi(x,y)|^2\mathbf{1}_{\{|\varphi(x,y)|\le 1\}} + c|\check{\varphi}(x,y)|^2\mathbf{1}_{\{|\check{\varphi}(x,y)|\le 1\}}$$

and so

$$J_t^{(2)}+\frac{1}{2}K_t^1=\sum_{s\le t}\psi^{(2)}(X_{s-},X_s)-\int_0^t\int_E\psi^{(2)}(X_s,y)N(X_s,dy)dH_s.$$

We then have, on $\{t<\zeta\}$,

$$e^{-\frac{1}{2}K_t^1}=\mathrm{Exp}(J_t^{(2)})\,e^{A_t^{\mu_2}},$$

where

$$\mu_2(dx):=\int_E\psi^{(2)}(x,y)N(x,dy)\mu_H(dx).$$

Let $J_t^{(3)}$ be the purely discontinuous locally square-integrable MAF on $[\![0,\zeta[\![$ with $\Delta J_t^{(3)}=\Delta J_t^{(1)}\Delta J_t^{(2)}$. Since $|e^x-1|\le|x|$, $x\in\mathbb{R}$, we obtain

$$J_t^{(3)}=\sum_{0<s\le t}\psi^{(3)}(X_{s-},X_s)-\int_0^t\int_E\psi^{(3)}(X_s,y)N(X_s,dy)dH_s,$$

where

$$\psi^{(3)}(x,y)=\left(\exp\left[-\frac{1}{2}\varphi(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le1\}}\right]-1\right)\times\left(\exp\left[\frac{1}{2}\varphi(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le1\}}+\frac{1}{2}\check{\varphi}(x,y)\mathbf{1}_{\{|\check{\varphi}(x,y)|\le1\}}\right]-1\right).$$

Thus

$$\begin{aligned}
e^{\Lambda(M)_t}&=e^{\Lambda(M^1)_t+A_t^\nu}\\
&=\exp\left(-\frac{1}{2}M_t^1-\frac{1}{2}M_t^1\circ r_t-\frac{1}{2}\varphi(X_t,X_{t-})\mathbf{1}_{\{|\varphi(X_t,X_{t-})|\le1\}}-\frac{1}{2}K_t^1\right)e^{A_t^\nu}\\
&=\mathrm{Exp}\left(\overline{M}_t+J_t^{(2)}+[\overline{M},J^{(2)}]_t\right)\cdot\mathrm{Exp}\left(\overline{M}_t\right)\circ r_t\\
&\qquad\cdot\exp\left(A_t^\nu+2A_t^{\mu_1}+A_t^{\mu_2}-\frac{1}{2}\varphi(X_t,X_{t-})\mathbf{1}_{\{|\varphi(X_t,X_{t-})|\le1\}}\right)\\
&=\mathrm{Exp}\left(\overline{M}_t+J_t^{(2)}+J_t^{(3)}\right)\cdot\mathrm{Exp}\left(\overline{M}_t\right)\circ r_t\\
&\qquad\cdot\exp\left(A_t^\nu+2A_t^{\mu_1}+A_t^{\mu_2}+A_t^{\mu_3}-\frac{1}{2}\varphi(X_t,X_{t-})\mathbf{1}_{\{|\varphi(X_t,X_{t-})|\le1\}}\right),
\end{aligned}$$

where $\mu_3(dx):=\int_E\psi^{(3)}(x,y)N(x,dy)\mu_H(dx)$. Here we use the exponential law for stochastic exponential. This says that the transform by $e^{\Lambda(M)}$ is a special case of the perturbation studied in Theorem 1.4. Theorem 1.7 now follows immediately from Theorem 1.4. □

Proof of Corollary 1.9.

Let $\{F_k\}$ be an $\mathcal{E}$-nest such that

$$\mathbf{1}_{F_k}\mu_{\langle M^u\rangle}\in\mathbf{K}_0(X)\qquad\text{for every } k\ge1.$$

It follows immediately from Theorem 1.7 that the corollary holds with the condition

$$\mathcal{Q}_{\alpha_0}(f,f) \geq 0 \qquad \text{for every } f \in \bigcup_k \mathcal{F}_{F_k,b}. \tag{2.2}$$

substituting for (1.13), and ($\mathcal{Q}$, $\bigcup_{k\geq 1} \mathcal{F}_{F_k,b}$) substituting for $(\mathcal{Q}, \mathcal{F}_b)$. It is clear that condition (1.13) implies (2.2). In fact these conditions are equivalent, because each $f \in \mathcal{F}_b$ can be approximated by a uniformly bounded sequence of functions in $\bigcup_{k\geq 1} \mathcal{F}_{F_k,b}$ in $\mathcal{E}_1^{1/2}$-norm; this implies that f^2 can be approximated by such a sequence. □

3 Examples

As a convention, for a measurable function f on E, we write $f \in \mathbf{K}(X)$ instead of $|f|dm \in \mathbf{K}(X)$. $f \in \mathbf{H}(X)$, $f \in \mathbf{K}_{loc}(X)$ are similarly defined.

Example 3.1 (Brownian motion) Let $X = (\Omega, X_t, \mathbf{P}_x)$ be d-dimensional Brownian motion on $\mathbb{R}^d$. The corresponding Dirichlet form $(\mathcal{E}, \mathcal{F}) = (\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ is given by

$$\begin{cases} H^1(\mathbb{R}^d) & := \{u \in L^2(\mathbb{R}^d) \mid \partial_i u \in L^2(\mathbb{R}^d) \text{ for } 1 \leq i \leq d\}, \\ \mathbf{D}(u,v) & := \displaystyle\int_{\mathbb{R}^d} \langle \nabla u(x), \nabla v(x) \rangle dx. \end{cases} \tag{3.1}$$

A signed Borel measure μ on $\mathbb{R}^d$ is said to be *of Kato class* if

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} \frac{|\mu|(dy)}{|x-y|^{d-2}} = 0 \qquad \text{when } d \geq 3,$$

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} (\log|x-y|^{-1})|\mu|(dy) = 0 \qquad \text{when } d = 2,$$

$$\sup_{x\in\mathbb{R}^d} \int_{|x-y|\leq 1} |\mu|(dy) < \infty \qquad \text{when } d = 1.$$

Here $|\mu| := \mu^+ + \mu^-$ is the total variation measure of μ.

A signed Borel measure μ on $\mathbb{R}^d$ is said to be *of local Kato class* if $\mathbf{1}_K\mu$ is of Kato class for evergy compact subset K of $\mathbb{R}^d$. By definition, any measure μ of local Kato class is always a signed Radon measure. It is proved in [1] that a positive measure μ is of the Kato class in the above sense if and only if $\mu \in \mathbf{K}(X)$.

Take measurable vector fields b, $\hat{b}$ and a measurable function c on $\mathbb{R}^d$ with $|b|^2 + |\hat{b}|^2 + |c| \in \mathbf{K}_{loc}(X) \cap \mathbf{H}(X)$ and assume $\sqrt{2\delta(|b|^2)} + \sqrt{2\delta(|\hat{b}|^2)} + \delta(|c|) \leq 1$. Here $\delta(|b|^2)$ is the coefficient of $\mathcal{E}$ for the $|b|^2 \in \mathbf{H}(X)$, other constants are similarly defined. Under these conditions, we can define a quadratic form $\mathcal{Q}$ on $C_0^\infty(\mathbb{R}^d)$ by

$$\begin{aligned} \mathcal{Q}(u,v) := \mathcal{E}(u,v) &- \int_{\mathbb{R}^d} v(x)\langle \nabla u(x), b(x) \rangle dx \\ &- \int_{\mathbb{R}^d} u(x)\langle \nabla v(x), \hat{b}(x) \rangle dx - \int_{\mathbb{R}^d} u(x)v(x)c(x)dx, \qquad u, v \in C_0^\infty(\mathbb{R}^d), \end{aligned} \tag{3.2}$$

Then the local martingale $M_t := \int_0^t b(X_s)dX_s$, (resp. $\widehat{M}_t := \int_0^t \hat{b}(X_s)dX_s$) with the Revuz measure $\mu_{\langle M\rangle}(dx) = |b|^2(x)dx$ (resp. $\mu_{\langle \widehat{M}\rangle}(dx) = |b|^2(x)dx$) of its quadratic process $\langle M\rangle_t = \int_0^t |b|^2(X_s)ds$ (resp. $\langle \widehat{M}\rangle_t = \int_0^t |b|^2(X_s)ds$) and CAF $C_t := \int_0^t c(X_s)ds$ satisfies the condition (1.7) for some $\alpha_0 \geq 0$.

In [7], we constructed a non-negative measurable function e on $\mathbb{R}^d$ with $e \in (\mathbf{K}_{loc}(X) \setminus \mathbf{K}(X)) \cap \mathbf{H}(X)$. We assume $b(x) = e^{1/2}(x)b^+$, $\hat{b}(x) = e^{1/2}(x)b^-$ and $c(x) \leq 0$ with $b^+, b^- \in \mathbb{R}^d$. Then $|b|^2, |\hat{b}|^2, |c| \in (\mathbf{K}_{loc}(X) \setminus \mathbf{K}(X)) \cap \mathbf{H}(X)$. Suppose $b^- = 0$. Then the condition (1.9) is satisfied by taking $\mathcal{D}_n$ as $L^\infty(F_n)$. So the conclusion of Theorem 1.4 holds for this case.

Next we suppose $b = \hat{b}$ and take $c = 0$. Then $(\mathcal{Q}, C_0^\infty(\mathbb{R}^d))$ has the following expression;

$$\mathcal{Q}(u,v) = \mathcal{E}(u,v) - \int_{\mathbb{R}^d} \langle \nabla(uv)(x), b(x)\rangle dx \quad \text{for } u,v \in C_0^\infty(\mathbb{R}^d). \tag{3.3}$$

In this case M satisfies the condition in Corollary 1.8.

Example 3.2 (Symmetric stable processes) We fix $\beta \in]0,2[$. Let $X = (\Omega, X_t, \mathbf{P}_x)_{x\in\mathbb{R}^d}$ be a Lévy process on $\mathbb{R}^d$ with

$$\mathbf{E}_0\left[e^{\sqrt{-1}\langle \xi, X_t\rangle}\right] = e^{-t|\xi|^\beta}.$$

X is called the *symmetric β-stable process.* Let $(\mathcal{E}, \mathcal{F})$ be the associated Dirichlet form on $L^2(\mathbb{R}^d)$ with X, which is given by

$$\begin{cases} \mathcal{E}(u,v) &= \dfrac{\mathcal{A}(d,-\beta)}{2} \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\beta}}dxdy, \\ \mathcal{F} &= \left\{u \in L^2(\mathbb{R}^d) \;\middle|\; \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\beta}}dxdy < \infty\right\}, \end{cases} \tag{3.4}$$

where

$$\mathcal{A}(d,\gamma) := \frac{|\gamma|\Gamma(\frac{d-\gamma}{2})}{2^{1+\gamma}\pi^{d/2}\Gamma(1+\frac{\gamma}{2})}, \quad \gamma < d.$$

X has a Lévy system (N, H), where $N(x,dy) := \mathcal{A}(d,-\beta)|x-y|^{-(d+\beta)}dy$ and $H_t = t$. So $\mu_H(dx) = dx$.

A signed Borel measure μ on $\mathbb{R}^d$ is said to be *of Kato class* if

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}\frac{|\mu|(dy)}{|x-y|^{d-\beta}} = 0 \qquad \text{when } d > \beta,$$

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}(\log|x-y|^{-1})|\mu|(dy) = 0 \qquad \text{when } d = \beta = 1,$$

$$\sup_{x\in\mathbb{R}^d}\int_{|x-y|\leq 1}|\mu|(dy) < \infty \qquad \text{when } d = 1 < \beta.$$

A Borel measure μ on $\mathbb{R}^d$ is said to be *of local Kato class* if $\mathbf{1}_K\mu$ is of Kato class for every compact subset K of $\mathbb{R}^d$. By definition, any measure μ of local Kato class is always a signed Radon measure.

It is proved in [11] (see [13] for the case $d > \beta$) that a positive measure μ is of the Kato class in the above sense if and only if $\mu \in \mathbf{K}(X)$.

In [7], we construct a non-negative measurable function e on $\mathbb{R}^d$ with $e \in (\mathbf{K}_{loc}(X) \setminus \mathbf{K}(X)) \cap \mathbf{H}(X)$ and $\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{e(y)}{|x-y|^{d-\beta}} dy < \infty$ (resp. $\sup_{x\in\mathbb{R}^d} \int_{|x-y|<e^{-1}} e(y) \log|x-y|^{-1} dy < \infty$) provided $d > \beta$ (resp. $d = \beta = 1$).

Take a non-negative Borel function φ on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal. Let M be locally square-integrable MAF with $\Delta M_t = \varphi(X_{t-}, X_t)$. For a constant $c > 0$, we set $\varphi(x,y) := c\sqrt{e(x)}\mathbf{1}_{\{|x-y|<1\}}|x-y|^\beta$ if $d > \beta$ and $\varphi(x,y) := c\sqrt{e(x)}\mathbf{1}_{\{|x-y|<e^{-1}\}}|x-y|^2 \log|x-y|^{-1}$ if $d = \beta = 1$. We see $\mu_{\langle M\rangle} = N(\varphi^2)d\mu_H \in (\mathbf{K}_{loc}(X) \setminus \mathbf{K}(X)) \cap \mathbf{H}(X)$. We can choose $c > 0$ such that (1.7) holds for some $\alpha_0 \geq 0$. Owing to the property of the function e on $\mathbb{R}^d$, we have $N(\check{\varphi}^2) \in L^\infty(\mathbb{R}^d)$, hence $N(\check{\varphi}^2) \in \mathbf{K}(X)$, which implies that the conditions in Corollary 1.8 hold. Then the following form $(\mathcal{Q}, C_0^\infty(\mathbb{R}^d))$ is well-defined: for $f, g \in C_0^\infty(\mathbb{R}^d)$.

$$\mathcal{Q}(f,g) \quad := \quad \mathcal{E}(f,g) + \mathcal{A}(d,-\beta) \int_{\mathbb{R}^d\times\mathbb{R}^d} (f(x)g(x) - f(y)g(y))\varphi(x,y) \frac{dxdy}{|x-y|^{d+\beta}}.$$

Then the symmetric semigroup $\overline{P}_t f(x) := \mathbf{E}_x\left[e^{\Lambda(M)_t} f(X_t)\right]$ is strongly continuous on $L^2(\mathbb{R}^d)$.

References

[1] M. Aizenman and B. Simon, Brownian motion and Harnack inequality for Schrödinger operators. *Comm. Pure. Appl. Math.* **35** (1982), no. 2, 209-273.

[2] S. Albeverio and Z.-M. Ma, Additive functionals, nowhere Radon measures and Kato class smooth measures associated with Dirichlet forms. *Osaka J. Math.* **29** (1992), no. 2, 247–265.

[3] S. Albeverio, F. Ru-Zong, M. Röckner and W. Stannat, A remark on coercive forms and associated semigroups. Partial differential operators and mathematical physics (Holzhau, 1994), 1–8, Oper. Theory Adv. Appl., **78**, Birkhäuser, Basel, 1995.

[4] Z.-Q. Chen, On Feynman-Kac perturbation of symmetric Markov processes. *Proceedings of Functional Analysis IX, Dubrovnik, Croatia, 2005,* pp. 39–43. Aarhus, 2007.

[5] Z.-Q. Chen, P. J. Fitzsimmons, K. Kuwae and T.-S. Zhang, Stochastic calculus for symmetric Markov processes. *Ann. Probab.* **36** (2008) 931–970.

[6] Z.-Q. Chen, P. J. Fitzsimmons, K. Kuwae and T.-S. Zhang, Perturbation of symmetric Markov processes. *Probab. Theory Relat. Fields* **140** (2008) 239–275.

[7] Z.-Q. Chen and K. Kuwae, On double Feller property. To appear in *Osaka J. Math.*, 2008.

[8] C.-Z. Chen, Z.-M. Ma and W. Sun, On Girsanov and generalized Feynman-Kac transforms for symmetric Markov processes. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** (2007), no. 2, 141–163.

[9] P.J. Fitzsimmons and K. Kuwae, Non-symmetric perturbations of symmetric Dirichlet forms. *J. Funct. Anal.* **208** (2004), no. 1, 140–162.

[10] H. Kunita, Sub-Markov semi-group in Banach lattices. *Proc. International Conf. on Func. Analysis Related Topics*. Tokyo (1969) 332-343.

[11] K. Kuwae and M. Takahashi, Kato class measuers of symmetric Markov processes under heat kernel estimates. *Jour. Func. Anal.* **250** (2007), no. 1, 86–113.

[12] S. Nakao, Stochastic calculus for continuous additive functionals of zero energy. *Z. Wahrsch. verw. Gebiete* **68** (1985), 557–578.

[13] Z. Zhao, A probabilistic principle and generalized Schrödinger perturbation. *J. Funct. Anal.* **101** (1991), no. 1, 162–17

Addresses:

Zhen-Qing Chen: Department of Mathematics, University of Washington, Seattle, WA 98195, USA. Email: `zchen@math.washington.edu`

Patrick J. Fitzsimmons: Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112, USA. Email: `pfitzsim@ucsd.edu`

Kazuhiro Kuwae: Department of Mathematics and Engineering, Graduate School of Science and Technology, Kumamoto University, Kumamoto 860-8555, Japan. Email: `kuwae@gpo.kumamoto-u.ac.jp`

Tusheng Zhang: School of Mathematics, University of Manchester, Sackville Street, Manchester M60 1QD, UK. Email: `tzhang@maths.manchester.ac.uk`