The Dirichlet Form of a Gradient-type Drift Transformation of a Symmetric Diffusion

by

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ABSTRACT

In the context of a symmetric diffusion process X admitting a *carré du champs* operator, we give a precise description of the Dirichlet form of the process obtained by subjecting X to a drift transformation of gradient type. This description relies on boundary-type conditions restricting an associated reflecting Dirichlet form.

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1. Introduction

Let $X = (X_t, \mathbf{P}^x)$ and $Y = (Y_t, \mathbf{Q}^x)$ be symmetric (*i.e.*, reversible) Markov diffusion processes with respective symmetry measures m and μ on a common state space E. Let ζ denote the lifetime of either process. It is shown in [**6**; §3] that if

$$\mathbf{Q}^\mu|_{\mathcal{F}_t\cap\{t<\zeta\}} \ll \mathbf{P}^m|_{\mathcal{F}_t\cap\{t<\zeta\}}, \qquad \forall t \ge 0, \tag{1.1}$$

then the associated Radon-Nikodym density process $L_t := d\mathbf{Q}^\mu|_{\mathcal{F}_t\cap\{t<\zeta\}}/d\mathbf{P}^m|_{\mathcal{F}_t\cap\{t<\zeta\}}$ has the form

$$L_t = \exp(M_t^\ell - \tfrac{1}{2}\langle M^\ell\rangle_t)1_{\{t<T_N\}}, \tag{1.2}$$

where N is an X-finely closed subset of E, $\ell := \frac{1}{2}\log(d\mu/dm)$ is locally in the Dirichlet space of (X, T_N) (the process X killed at the hitting time T_N of N), and M^ℓ is the local martingale continuous additive functional of (X, T_N) associated with ℓ by Fukushima's decomposition.

Conversely, given an X-finely closed set N_0 and a function ℓ locally in the Dirichlet space of (X, T_{N_0}), we can set $N = N_0 \cup \{\tilde{\ell} \notin \mathbf{R}\}$, and then use the right side of (1.2) to define a process $L = (L_t)$. (Here and in what follows, the tilde indicates that a quasi-continuous m-version of the function has been selected.) It is standard that L so defined is a supermartingale multiplicative functional of X and a local martingale on $[0, T_N[$. The formulae $\mathbf{Q}^x|_{\mathcal{F}_t\cap\{t<\zeta\}} := L_t \cdot \mathbf{P}^x|_{\mathcal{F}_t\cap\{t<\zeta\}}$, $x \in E \setminus N$, then determine the law of a diffusion Y with state space $E \setminus N$ and symmetry measure $\mu := \psi^2 \cdot m$, where $\psi := \exp(\ell)$. Evidently, $\mathbf{Q}^\mu \ll_{\rm loc} \mathbf{P}^m$ in the sense of (1.1).

The Dirichlet forms $(\mathcal{E}^X, \mathcal{D}^X)$ and $(\mathcal{E}^Y, \mathcal{D}^Y)$ of X and Y are easily related, at least formally. To this end recall that with each pair (u, v) of elements of $\mathcal{D}^X$ one can associate a signed measure $\mu^X_{\langle u,v\rangle}$ (the *mutual energy measure* of u and v) such that the mapping $(u, v) \mapsto \mu^X_{\langle u,v\rangle}$ is symmetric and bilinear, and

$$\mathcal{E}^X(u, v) = \tfrac{1}{2}\mu^X_{\langle u,v\rangle}(E), \qquad \forall u, v \in \mathcal{D}^X. \tag{1.3}$$

As is customary, we write $\mu_{\langle u\rangle}$ for $\mu_{\langle u,u\rangle}$. Now let (G_n) be an increasing sequence of X-finely open subsets of $E \setminus N$ such that (i) $E \setminus N$ differs from $\cup_n G_n$ by an X-polar set, (ii) $\tilde{\ell}|_{G_n}$ is bounded for all $n \in \mathbf{N}$, and (iii) $\mu_{\langle \ell\rangle}(G_n) < \infty$ for all n. (The properties of ℓ noted earlier ensure that $\mu_{\langle \ell\rangle}$ can be defined on Borel subsets of $E \setminus N$ by a localization argument. These same properties imply the existence of a sequence (G_n) with the listed properties.) It follows from [**6**; (4.9)] that the vector space

$$\mathcal{C} := \cup_n \{u \in \mathcal{D}^X \cap L^\infty(m) : \tilde{u} = 0 \text{ quasi-everywhere on } E \setminus G_n\} \tag{1.4}$$

is dense in $\mathcal{D}^Y$ and that

$$\mathcal{E}^Y(u, v) = \tfrac{1}{2}\int_{E\setminus N} \tilde{\psi}^2 \, d\mu^X_{\langle u,v\rangle}, \qquad \forall u, v \in \mathcal{C}. \tag{1.5}$$

In fact, $\mu^Y_{\langle u,v\rangle} = \tilde\psi^2 \mu^X_{\langle u,v\rangle}$ for $u, v \in \mathcal{C}$.

It is desirable to specify a "core" for $\mathcal{D}^Y$ in simpler terms; *e.g.*, without reference to a "nest" such as (G_n). The work of Eberle [**3**] shows that this is indeed possible under some restrictions on ℓ, at least when X admits a *carré du champs* operator. Our goal in this paper is to extend and amplify the results of Eberle, in a general context. A special case of our main result can be stated as follows. Let N and ℓ be as in the first paragraph of this introduction, and put $\psi = \exp(\ell)$ as before. Recall that the symmetry measure of Y is $\mu = \psi^2 \cdot m$. Fix $0 < p < \infty$, define

$$\mathcal{C}^\psi_p := \{u \in \mathcal{D}^X \cap L^\infty(m) : u \in L^2(\psi^2 \cdot m), \tilde\psi \in L^2(\mu^X_{\langle u\rangle}), \tilde u \in L^p(\mu^X_{\langle \psi\rangle})\}, \tag{1.6}$$

and let $\mathcal{E}^\psi(u,v)$ be defined on $\mathcal{C}^\psi_p$ by the right side of (1.5). We shall prove that the symmetric bilinear form $(\mathcal{E}^\psi, \mathcal{C}^\psi_p)$ is closable, and that its closure is precisely $(\mathcal{E}^Y, \mathcal{D}^Y)$.

The condition $\tilde u \in L^p(\mu^X_{\langle\psi\rangle})$ in (1.6) may be thought of as an approximate boundary condition. Suppressing this condition we obtain $\mathcal{C}^\psi_0 := \{u \in \mathcal{D}^X \cap L^\infty(m) : u \in L^2(\psi^2 \cdot m), \tilde\psi \in L^2(\mu^X_{\langle u\rangle})\}$. The right side of (1.5) still serves to define $\mathcal{E}^\psi$ on $\mathcal{C}^\psi_0$; the form so defined is closable and its closure $(\mathcal{E}^\psi_0, \mathcal{D}^\psi_0)$ is an extension of $(\mathcal{E}^Y, \mathcal{D}^Y)$. In particular, $\mathcal{D}^Y \subset \mathcal{D}^\psi_0$, and this containment can be strict as we see by taking X to be the 3-dimensional Bessel process on $E =]0,\infty[$ and Y the absorbed Brownian motion on E. See (3.9) for details.

The work of [**6**] has been extended to general symmetric Markov processes in [**2**], and it seems likely that results similar to those found in this paper can be obtained in that context.

In section 2 we describe in detail the setting of the paper and we record some preliminary results. Section 3 contains the proof of the main result outlined above.

2. Preliminaries

Let $(E, \mathcal{B}(E))$ be a Lusin metrizable topological space; *i.e.*, E is homeomorphic to a Borel subset of some compact metric space and $\mathcal{B}(E)$ is the class of Borel sets in E. Let m be a σ-finite measure on E and let $(\mathcal{E}, \mathcal{D})$ be a quasi-regular [**13**; IV.3] Dirichlet form on $L^2(m)$. We assume that $(\mathcal{E}, \mathcal{D})$ is *strongly local* in the sense that

$$F, G \in C^\infty_c(\mathbf{R}), F \text{ constant on the support of } G \quad \Longrightarrow \quad \mathcal{E}(F_0 \circ u, G_0 \circ u) = 0,\ \forall u \in \mathcal{D}, \tag{2.1}$$

where $F_0 := F - F(0), G_0 := G - G(0)$. Under these conditions there is a right Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbf{P}^x)$ that is symmetric with respect to m and properly associated with $(\mathcal{E}, \mathcal{D})$. Consequently, if $(P_t)_{t\ge 0}$ denotes the transition semigroup of X, then

$$(f, P_t g)_m = (P_t f, g)_m \qquad \forall f, g \in L^2(m), \tag{2.2}$$

where $(u,v)_m := \int_E uv\, dm$ is the natural inner product in $L^2(m)$. Viewed as a semigroup operating in $L^2(m)$, (P_t) is strongly continuous, and $P_t f$ is $\mathcal{B}(E)$-measurable for each bounded $\mathcal{B}(E)$-measurable function $f\colon E \to \mathbf{R}$. Conditions (2.1) and (2.2) imply that X is a *diffusion* in the following strong sense:

(2.3)(i) The $\mathbf{P}^m$-completion $(\mathcal{F}_t)_{t\geq 0}$ of the natural filtration $\sigma\{X_s; 0 \leq s \leq t\}$, $t \geq 0$, is quasi-left-continuous and the lifetime of X, denoted ζ, is an $(\mathcal{F}_t)$ *predictable* stopping time;

(2.3)(ii) $t \mapsto X_t$ is continuous on $[0, \zeta\,[$, $\mathbf{P}^m$-a.s.

Consequently, every $(\mathcal{F}_t)$-stopping time is predictable, and every $(\mathcal{F}_t)$-martingale has continuous paths ($\mathbf{P}^m$-a.s.). See [**14**; §47] and [**13**; IV.3].

We can (and do) take the sample space Ω to be the space of paths ω from $[0, \infty[$ to $E \cup \{\Delta\}$ that are E-valued and continuous on $[0, \zeta(\omega)[$ and that hold the value $\Delta \notin E$ after time $\zeta(\omega)$. As usual, any function f defined on E is automatically extended to the cemetery state Δ by the convention $f(\Delta) = 0$.

The Dirichlet form $(\mathcal{E}, \mathcal{D})$ is related to the transition semigroup (P_t) by

$$\mathcal{D} = \{u \in L^2(m) : \sup_{t>0} \frac{1}{t}(u, u - P_t u)_m < \infty\};$$

$$\mathcal{E}(u, v) = \lim_{t\to 0} \frac{1}{t}(u, v - P_t v)_m, \qquad u, v \in \mathcal{D}.$$

Endowed with the inner product $\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + (u, v)_m$, $\mathcal{D}$ is a Hilbert space.

Each element $u \in \mathcal{D}$ admits an m-modification $\tilde{u}$ (a quasi-continuous version) such that $t \mapsto \tilde{u}(X_t)$ is continuous on $[0, \infty\,[$, $\mathbf{P}^m$-a.s. We then have Fukushima's decomposition [**8**; Thm. 5.2.2]:

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u, \quad \forall t \geq 0, \mathbf{P}^x\text{-a.s. for q.e. } x \in E, \tag{2.4}$$

where M^u and N^u are continuous additive functionals (CAFs) of X, M^u is a martingale such that $\sup_{t>0} t^{-1}\mathbf{P}^m[(M_t^u)^2] < \infty$, and $\lim_{t\to 0} t^{-1}\mathbf{P}^m[(N_t^u)^2] = 0$. This decomposition is unique. In the sequel we shall refer to M^u as the martingale part of u.

Let $\tau(G)$ denote the first exit time from $G \subset E$; that is, $\tau(G) := \inf\{t > 0 : X_t \notin G\}$. A function $u: E \to \mathbf{R}$ is locally in $\mathcal{D}$ (notation: $u \in \mathcal{D}_{\text{loc}}$) provided there is an increasing sequence $\{G_n\}$ of finely open sets with $\lim_n \tau(G_n) = \zeta$, $\mathbf{P}^m$-a.s., and a sequence $\{u_n\}$ of elements of $\mathcal{D}$ such that $u = u_n$ a.e.-m on G_n for each $n \in \mathbf{N}$. Each $u \in \mathcal{D}_{\text{loc}}$ admits a quasi-continuous modification $\tilde{u}$ such that (2.4) holds for $t \in [0, \zeta\,[$, where now M^u is a local martingale CAF and N^u is a CAF locally of zero energy. See [**8**; §5.5]. For convenience, *in the sequel we will always take* $u \in \mathcal{D}_{\text{loc}}$ *to be represented by its quasi-continuous version,* and will drop the $\tilde{}$ from the notation.

Given $u \in \mathcal{D}_{\text{loc}}$, the local martingale CAF M^u admits a quadratic variation process $\langle M^u \rangle$; *i.e.*, $\langle M^u \rangle$ is predictable and $(M^u)^2 - \langle M^u \rangle$ is a local martingale on $[0, \zeta[$. Since X is a diffusion, $\langle M^u \rangle$ is a CAF. The Revuz measure of $\langle M^u \rangle$ (the so-called energy measure of u) is the smooth measure $\mu_{\langle u \rangle}$ on E determined by

$$\mu_{\langle u \rangle}(f) = \uparrow \lim_{t\to 0} t^{-1}\mathbf{P}^m \int_0^t f(X_s)\, d\langle M^u \rangle_s, \qquad f \in p\mathcal{B}(E). \tag{2.5}$$

If $u \in \mathcal{D}$ then $\mu_{\langle u \rangle}$ has finite total mass; indeed,

$$\mathcal{E}(u, u) = \frac{1}{2}\mu_{\langle u \rangle}(E), \qquad u \in \mathcal{D}.$$

Given two elements u and v of $\mathcal{D}_{\text{loc}}$, the quadratic covariation $\langle M^u, M^v\rangle$ is a (signed) CAF of X and since $\frac{1}{4}[\langle M^{u+v}\rangle - \langle M^{u-v}\rangle]$, we have

$$\mu_{\langle u,v\rangle} = \tfrac{1}{4}[\mu_{\langle u+v\rangle} - \mu_{\langle u-v\rangle}]. \tag{2.6}$$

(This formula can be taken literally if both u and v lie in $\mathcal{D}$. In general there is a sequence $\{G_n\}$ of finely open sets with $\mathbf{P}^m[\lim_n \tau(G_n) < \zeta] = 0$, and sequences $\{u_n\}, \{v_n\}$ from $\mathcal{D}$ with $u = u_n$ and $v = v_n$ m-a.e. on G_n for each n; things being so, the two sides of (2.6) determine the same finite signed measure on each G_n.) We then have

$$\mathcal{E}(u,v) = \tfrac{1}{2}\mu_{\langle u,v\rangle}(E), \qquad \forall u,v \in \mathcal{D}. \tag{2.7}$$

It can be shown [**6**; §5] that $(u,v) \mapsto \mu_{\langle u,v\rangle}$ is a continuous mapping of $\mathcal{D}\times\mathcal{D}$ into the space of finite signed measures on E endowed with the total variation norm. (Convergence in $\mathcal{D}$ is with respect to the norm $(\mathcal{E}_1)^{1/2}$.) See [**8**; Ch. 1] for further details.

If $F : \mathbf{R} \to \mathbf{R}$ is locally absolutely continuous with $F' \in L^\infty_{\text{loc}}(\mathbf{R})$ and if $u \in \mathcal{D}_{\text{loc}}$, then $F\circ u \in \mathcal{D}_{\text{loc}}$ and by Itô's formula $\langle M^{F\circ u}\rangle_t = \int_0^t [F'\circ u]^2(X_s)\, d\langle M^u\rangle_s$. From this and (2.5) it follows that

$$\mu_{\langle F\circ u\rangle} = [F'\circ u]^2 \mu_{\langle u\rangle}. \tag{2.8}$$

Polarizing the special case $F(t) = t^2$ of (2.8), we see that if $u, v \in \mathcal{D}_{\text{loc}}$ (in which case $uv \in \mathcal{D}_{\text{loc}}$), then $\mu_{\langle uv\rangle} = u^2\mu_{\langle v\rangle} + 2uv\mu_{\langle u,v\rangle} + v^2\mu_{\langle u\rangle}$. Also, the positivity of $\mu_{\langle u+tv\rangle} = \mu_{\langle u\rangle} + 2t\mu_{\langle u,v\rangle} + t^2\mu_{\langle v\rangle}$ for all $t \in \mathbf{R}$ implies that $|\mu_{\langle u,v\rangle}(f)| \le \mu_{\langle u\rangle}(f)^{1/2}\mu_{\langle v\rangle}(f)^{1/2}$ for any $f \in L^1(\mu_{\langle u\rangle} + \mu_{\langle v\rangle})$. These remarks yield the useful estimate:

$$\mu_{\langle uv\rangle} \le 2[u^2\mu_{\langle v\rangle} + v^2\mu_{\langle u\rangle}], \qquad u, v \in \mathcal{D}_{\text{loc}}. \tag{2.9}$$

We shall use $\mathcal{M}$ to denote the class of martingale CAFs of finite energy. More precisely, define the mutual energy $e(M, M')$ of two martingale CAFs M and M' by

$$e(M,M') := \lim_{t\downarrow 0}(2t)^{-1}\mathbf{P}^m[M_t M'_t] = \lim_{t\downarrow 0}(2t)^{-1}\mathbf{P}^m\langle M, M'\rangle_t,$$

whenever the limit exists, and write $e(M) = e(M,M)$. Then

$$\begin{aligned}\mathcal{M} := \{M :&M \text{ is a CAF of } X,\ \mathbf{P}^x[M_t^2] < \infty \text{ and}\\ &\mathbf{P}^x[M_t] = 0 \text{ for quasi-every } x \in E,\ e(M) < \infty\}.\end{aligned}$$

When equipped with the inner product $(M, M') \mapsto e(M, M')$, $\mathcal{M}$ becomes a Hilbert space; the associated norm is $e(\cdot)^{1/2}$.

It can be shown that the Dirichlet form $(\mathcal{E}_{\text{ref}}, \mathcal{D}_{\text{ref}})$ discussed below coincides with the *active reflected Dirichlet from* associated with $(\mathcal{E}, \mathcal{D})$ as in [**1**]. Since we shall only need the fact that the active reflected Dirichlet space is an extension of $(\mathcal{E}_{\text{ref}}, \mathcal{D}_{\text{ref}})$ (and this assertion is obvious from the definitions), we shall not prove this claim. The reader is referred to [**1**] and [**10**] for more on these matters.

(2.10) Proposition. *The form defined by*

$$\mathcal{D}_{\rm ref} := \{u \in \mathcal{D}_{\rm loc} \cap L^2(m) : \mu_{\langle u \rangle}(E) < \infty\},$$

$$\mathcal{E}_{\rm ref}(u,v) := \tfrac{1}{2}\mu_{\langle u,v \rangle}(E), \qquad u, v \in \mathcal{D}_{\rm ref},$$

is a strongly local Dirichlet form on $L^2(m)$. Moreover, $b\mathcal{D}$ is an algebra ideal in $b\mathcal{D}_{\rm ref}$.

Proof. Let us first check that $(\mathcal{E}_{\rm ref}, \mathcal{D}_{\rm ref})$ is a Dirichlet form in $L^2(m)$. For this it suffices to check that $(\mathcal{E}_{\rm ref}, \mathcal{D}_{\rm ref})$ is closed. Let $\{u_n\} \subset \mathcal{D}_{\rm ref}$ be an $(\mathcal{E}_{\rm ref})_1$-Cauchy sequence with $u_n \to u$ in $L^2(m)$. Let M^n denote the (local) martingale part in the Fukushima decomposition of u_n. The quadratic variation of M^n satisfies

$$\mathbf{P}^m[\langle M^n \rangle_t] \le 2t\mathcal{E}_{\rm ref}(u_n, u_n),$$

from which it follows that $M^n \in \mathcal{M}$. Moreover, $e(M^n - M^k) = \mathcal{E}_{\rm ref}(u_n - u_k, u_n - u_k)$ so that (M^n) is a Cauchy sequence in $\mathcal{M}$. Thus, there exists $M \in \mathcal{M}$ such that $e(M^n - M) \to 0$ as $n \to \infty$. According to [**8**; Thm. 5.2.1], this means that $\sup_{0 \le s \le t} |M^n_s - M_s| \to 0$ as $n \to \infty$, in $L^2(\mathbf{P}^m)$, for all $t > 0$. Now the Lyons-Zheng decomposition [**12, 11**], as developed in [**5**], asserts that

$$u_n(X_t) - u_n(X_0) = [M^n_t - M^n_t \circ r_t]/2, \qquad \mathbf{P}^m\text{-a.s. on } \{t < \zeta\},\ \forall t > 0. \tag{2.11}$$

Here, for $t > 0$, $r_t : \{t < \zeta\} \to \Omega$ denotes the time-reversal operator on $[0,t]$:

$$r_t\omega(s) = \begin{cases} \omega(t-s), & 0 \le s \le t; \\ \omega(0), & s > t. \end{cases}$$

[The m-symmetry of X is equivalent to the assertion that for each $t > 0$, r_t is a measure-preserving transformation of the measure space $(\{t < \zeta\}, \mathcal{F}_t \cap \{t < \zeta\}, \mathbf{P}^m[\,\cdot\,; t < \zeta)]$.] Passing to the limit in (2.11) (along a subsequence if necessary), using the measure-preserving property of r_t, we find that for fixed $t > 0$,

$$u(X_t) - u(X_0) = [M_t - M_t \circ r_t]/2, \qquad \text{on } \{t < \zeta\}, \mathbf{P}^m\text{-a.s.} \tag{2.12}$$

Using [**6**; (4.15)] we can deduce from (2.12) that $u \in \mathcal{D}_{\rm loc}$ and that M is the martingale part in the Fukushima decomposition of u. This yields $\mu_{\langle u \rangle}(E) = 2e(M) < \infty$ so that $u \in \mathcal{D}_{\rm ref}$. Thus, $(\mathcal{E}_{\rm ref}, \mathcal{D}_{\rm ref})$ is closed.

To check the asserted ideal property, we need to show that $gu \in \mathcal{D}$ whenever $g \in b\mathcal{D}$ and $u \in b\mathcal{D}_{\rm ref}$. But for $t > 0$ we have

$$\begin{aligned} t^{-1}(gu, gu - P_t(gu))_m = (2t)^{-1}&\mathbf{P}^m[(gu(X_t) - gu(X_0))^2; t < \zeta] \\ &+ t^{-1}\mathbf{P}^m[(gu(X_0))^2; \zeta \le t]. \end{aligned} \tag{2.13}$$

Since u is bounded, the second term on the right side of (2.13) is dominated by a multiple of $t^{-1}\mathbf{P}^m[(g(X_0))^2; \zeta \le t]$, which remains bounded as $t \to 0$ because $g \in \mathcal{D}$. To treat the first term on the right in (2.13), note that $gu \in \mathcal{D}_{\rm loc}$. The Lyons-Zheng decomposition now yields the estimate

$$(2t)^{-1}\mathbf{P}^m[(gu(X_t) - gu(X_0))^2; t < \zeta] \le C\mu_{\langle gu \rangle}(E) \le 2C\left[\|u\|_\infty^2 \mathcal{E}(g,g) + \|g\|_\infty^2 \mathcal{E}_{\rm ref}(u,u)\right] < \infty.$$

Thus, $\sup_{t>0} t^{-1}(gu, gu - P_t(gu))_m < \infty$. Since $gu \in L^2(m)$, we have shown that $gu \in \mathcal{D}$. ∎

3. Drift Transformations

Our first task is to discuss the Girsanov transformation of X induced by $\psi \in p\mathcal{D}^X_{\rm loc}$. Fix $\psi \in p\mathcal{D}^X_{\rm loc}$ and let N_0 be a properly exceptional set for (X, m) ([**8**; p. 134]) such that the Fukushima decomposition

$$\psi(X_t) - \psi(X_0) = M^\psi_t + N^\psi_t, \qquad \forall t \ge 0, \mathbf{P}^x\text{-a.s.}, x \in E \setminus N_0, \tag{3.1}$$

holds. We can (and do) assume that ψ is finely continuous and finite-valued on $E \setminus N_0$. Define $N_1 := \{x \in E \setminus N_0 : \psi(x) = 0\}$ and $N = N_\psi := N_0 \cup N_1$. Then N is X-finely closed, and $\ell := \log \psi$ is an element of $(\mathcal{D}^X_{E\setminus N})_{\rm loc}$, so we have the (local) Fukushima decomposition

$$\ell(X_t) - \ell(X_0) = M^\ell_t + N^\ell_t, \qquad \forall 0 \le t < T_N, \mathbf{P}^x\text{-a.s.}, x \in E \setminus N, \tag{3.2}$$

where $T_N := \inf\{t > 0 : X_t \in N\}$ is the hitting time of N, M^ℓ is a local martingale CAF of (X, T_N) and N^ℓ is a CAF of (X, T_N) locally of zero energy. Here (X, T_N) denotes the process X killed at time T_N and $(\mathcal{E}^X_{E\setminus N}, \mathcal{D}^X_{E\setminus N})$ is the associated Dirichlet space. The stochastic exponential $L_t := \exp(M^\ell_t - \frac{1}{2}\langle M^\ell\rangle_t)1_{t<T_N}$ of M^ℓ is a (positive) supermartingale multiplicative functional of (X, T_N) and a local martingale on $[0, T_N[$. By standard theory (e.g. [**14**; § 62]) there is a family $(\mathbf{Q}^x)_{x\in E} = (\mathbf{Q}^x_\psi)_{x\in E\setminus N}$ of probability laws on $(\Omega, \mathcal{F}, \mathcal{F}_t)$ such that

$$\mathbf{Q}^x(F; t < \zeta) = \mathbf{P}^x[F \cdot L_t; t < \zeta], \qquad F \in b\mathcal{F}_t, t \ge 0,\ x \in E \setminus N. \tag{3.3}$$

For emphasis we use $(Y_t)_{t\ge 0}$ to denote the coordinate process under the $\mathbf{Q}^x$.

Let μ denote the measure $\psi^2 \cdot m$. The process $Y := (Y_t, \mathbf{Q}^x)$ is a μ-symmetric diffusion with state space $E_\psi := E \setminus N_\psi$; see, for example, [**5**; § 4]. We refer to Y as the Girsanov transformation of X based on ψ. Let $(\mathcal{E}^Y, \mathcal{D}^Y)$ denote the Dirichlet form of Y. This is a strongly local Dirichlet form on $L^2(\mu) = L^2(\mu, E_\psi)$.

The following summary is drawn from [**6**; §4].

(3.4) Proposition. *Fix $\psi \in p\mathcal{D}^X_{\rm loc}$ and write N for N_ψ.*

(a) Let G be an X-finely open subset of $E \setminus N$ on which ψ is bounded away from zero and infinity. Furthermore, assume $\mu_{\langle\psi\rangle}(G) < \infty$. Then $\mathcal{D}^Y_G = \mathcal{D}^X_G$, and if $u, v \in \mathcal{D}^Y_G$ then $\mathcal{E}^Y(u, v) = \frac{1}{2}\int_G \psi^2\, d\mu_{\langle u\rangle}$.

(b) Let $\{G_n\} \subset E$ be an increasing sequence of X-finely open subsets of $E \setminus N$ such that (i) $(E \setminus N) \setminus \cup_n G_n$ is (X, m)-polar, (ii) $\mu_{\langle\psi\rangle}(G_n) < \infty$ for all $n \in \mathbf{N}$, and (iii) there are constants $0 < C_n < \infty$ such that $C_n^{-1} \le \psi^2 \le C_n$ on G_n, $\forall n$. Then $\cup_n \mathcal{D}^X_{G_n}$ is $\mathcal{E}^Y_1$-dense in $\mathcal{D}^Y$, and

$$\mathcal{E}^Y(u, u) = \frac{1}{2}\int_E \psi^2\, \mu_{\langle u\rangle}, \qquad \forall u \in \cup_n \mathcal{D}^X_{G_n}. \tag{3.5}$$

(c) $\mathcal{D}^Y_{\rm loc} = (\mathcal{D}^X_{E_\psi})_{\rm loc}$, and $\mu^Y_{\langle u\rangle} = \psi^2 \mu_{\langle u\rangle}$ for all $u \in \mathcal{D}^Y_{\rm loc}$.

(d) Any (X, T_N)-nest is a Y-nest.

Of course, to apply (3.4)(b) we require the existence of a sequence (G_n) as specified there. This matter is addressed in the following

(3.6) Lemma. *If $\psi \in p\mathcal{D}^X_{\text{loc}}$, then there is a sequence (G_n) of X-finely open sets satisfying the conditions listed in (3.4)(b).*

Proof. Since $\psi \in \mathcal{D}^Y_{\text{loc}}$, the measure $\mu_{\langle\psi\rangle}$ is smooth; indeed it is the smooth measure corresponding to the CAF $\langle M^\psi\rangle$. By a standard construction [**8**; Lemma 5.1.7], there is a bounded 1-excessive function $f \in \mathcal{D}$ such that $\{f = 0\}$ is X-exceptional and $\mu_{\langle\psi\rangle}(f) < \infty$. The set $H_n := \{f > 1/n\}$ evidently satisfies $\mu_{\langle\psi\rangle}(H_n) < \infty$, and $E \setminus \cup_n H_n = \{f = 0\}$. Consequently, the sequence (G_n) defined by $G_n := H_n \cap \{n^{-1} < \psi < n\}$ has the required properties. ∎

We now introduce the vector space

$$\mathcal{C}^\psi_0 := \{u \in \mathcal{D}^X \cap L^\infty(m) : u \in L^2(\mu), \psi \in L^2(\mu_{\langle u\rangle})\}$$

on which is defined the symmetric form

$$\mathcal{E}^\psi(u,v) := \tfrac{1}{2}\int_{E_\psi} \psi^2 \, d\mu_{\langle u,v\rangle}. \tag{3.7}$$

In view of (2.8), normalized contractions operate on $(\mathcal{E}^\psi, \mathcal{C}^\psi_0)$, and the form $(\mathcal{E}^\psi, \mathcal{C}^\psi_0)$ is strongly local.

(3.8) Proposition. *Fix $\psi \in p\mathcal{D}^X_{\text{loc}}$ and let the form $(\mathcal{E}^\psi, \mathcal{C}^\psi_0)$ be as defined above. Let Y denote the Girsanov transformation of X based on ψ, with associated Dirichlet form $(\mathcal{E}^Y, \mathcal{D}^Y)$.*

(a) The form $(\mathcal{E}^\psi, \mathcal{C}^\psi_0)$ is closable in $L^2(\mu, E_\psi)$, and its closure $(\mathcal{E}^\psi_0, \mathcal{D}^\psi_0)$ is a strongly local Dirichlet form. Moreover, $\mathcal{D}^\psi_0 \subset \mathcal{D}^Y_{\text{loc}}$ and $\mathcal{E}^\psi_0 = \mathcal{E}^\psi$ on $\mathcal{D}^\psi_0 \times \mathcal{D}^\psi_0$.

(b) $\mathcal{D}^Y \subset \mathcal{D}^\psi_0$, $\mathcal{E}^Y = \mathcal{E}^\psi$ on $\mathcal{D}^Y \times \mathcal{D}^Y$, and $b\mathcal{D}^Y$ is an algebra ideal in $b\mathcal{D}^\psi_0$.

Proof. (a) By (3.4)(c) and (2.11) applied to Y, the form $\mathcal{E}^\psi$ on the domain

$$\mathcal{D}^Y_{\text{ref}} := \{u \in \mathcal{D}^Y_{\text{loc}} \cap L^2(\mu) : \psi \in L^2(\mu_{\langle u\rangle})\}$$

is closed. Its restriction $(\mathcal{E}^\psi, \mathcal{C}^\psi_0)$ is therefore closable. Because of [**8**; Thm. 3.1.1], the closure $(\mathcal{E}^\psi_0, \mathcal{D}^\psi_0)$ of $(\mathcal{E}^\psi, \mathcal{C}^\psi_0)$ is a Dirichlet form. Also, since $(\mathcal{E}^\psi, \mathcal{D}^Y_{\text{ref}})$ is closed, $\mathcal{D}^\psi_0 \subset \mathcal{D}^Y_{\text{ref}} \subset \mathcal{D}^Y_{\text{loc}}$. For the same reason, $\mathcal{E}^\psi_0 = \mathcal{E}^\psi$ on $\mathcal{D}^\psi_0 \times \mathcal{D}^\psi_0$. Finally, $\mathcal{E}^\psi_0$ is strongly local by virtue of (3.7) and the polarization of (2.8).

(b) The inclusion $\mathcal{D}^Y \subset \mathcal{D}^\psi_0$ is an obvious consequence of (3.4)(b). The second assertion follows immediately from (2.10) since $\mathcal{D}^\psi_0 \subset \mathcal{D}^Y_{\text{ref}}$ by part (a). ∎

The significance of (3.8)(b) is this: According to [**15**; Thm. 20.1], there is a quasi-regular representation $(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ of $(\mathcal{E}^\psi, \mathcal{D}^\psi_0)$ with state space $\tilde{E}$ such that E_ψ is quasi-open in $\tilde{E}$, and such that if $\tilde{Y} = (\tilde{Y}_t, \tilde{\mathbf{Q}}^x)$ is the symmetric process associated with $(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ then $\tilde{Y}$ killed at the first exit time from E_ψ coincides (in law) with Y. Consequently, $\mathcal{D}^Y$ may be identified with the set of elements of $\tilde{\mathcal{D}}$ that vanish on $\tilde{E} \setminus E_\psi$. Our goal is to give a more concrete description of $\mathcal{D}^Y$ through the use of what might be called "pseudo boundary conditions".

The reader will find in [**3**] conditions under which $\mathcal{D}^Y = \mathcal{D}_0^\psi$. Roughly speaking, the class of drifts admitted for consideration in [**3**] is restricted to those for which localization is needed only at "metric infinity". Here is an example to show that the inclusion $\mathcal{D}^Y \subset \mathcal{D}_0^\psi$ may be strict.

(3.9) Example. Let X be a three-dimensional Bessel process; that is, X is the radial part of a three dimensional Brownian motion. We take the state space of X to be $]0, +\infty[$. The associated Dirichlet form is

$$\mathcal{E}^X(u, v) = \tfrac{1}{2} \int_0^\infty u'(x)v'(x)\, x^2\, dx$$

acting on the class $\mathcal{D}^X$ of all locally absolutely continuous functions $u\colon]0, \infty[\to \mathbf{R}$ such that both u and u' are square integrable with respect to the symmetry measure $m(dx) = x^2\, dx$. Now take $\psi(x) = 1/x \in \mathcal{D}_{\rm loc}^X$. In this case the process Y is just the absorbed Brownian motion on $]0, \infty[$, so $\mathcal{D}^Y$ is the class of all locally absolutely continuous functions $u\colon]0, \infty[\to \mathbf{R}$ such that (i) $\lim_{x \downarrow 0} u(x) = 0$ and (ii) both u and u' are square integrable with respect to the symmetry measure $\mu(dx) := dx$. Meanwhile, $\mathcal{D}_0^\psi$ consists of the class obtained when the boundary condition (i) above is dropped. The inclusion $\mathcal{D}^Y \subset \mathcal{D}_0^\psi$ is strict—if $u\colon [0, \infty[\to \mathbf{R}$ is any smooth function of compact support and if u is constant on some interval of the form $[0, b[$, then the restriction of u to $]0, \infty[$ is an element of $\mathcal{D}_0^\psi \setminus \mathcal{D}^Y$. $\square$

We now introduce certain vector subspaces of $\mathcal{C}_0^\psi$. Let Σ denote the class of function σ : $[0, \infty[\to [0, \infty[$ such that (i) σ is increasing, (ii) $\sigma(t) > 0$ for $t > 0$, (iii) $\sigma(s + t) \le C_\sigma[\sigma(s) + \sigma(t)]$ for all $s, t \ge 0$ and some constant $0 < C_\sigma < \infty$. Now define, for $\psi \in p\mathcal{D}_{\rm loc}^X$ and $\sigma \in \Sigma$,

$$\mathcal{C}_\sigma^\psi := \{u \in \mathcal{C}_0^\psi : \sigma(|u|) \in L^1(\mu_{\langle\psi\rangle})\}.$$

Properties (i) and (iii) imply that $\mathcal{C}_\sigma^\psi$ is a vector space closed under composition with normalized contractions. It is easy to see that

$$\mathcal{C}_{0+}^\psi := \cup_{\sigma \in \Sigma} \mathcal{C}_\sigma^\psi = \{u \in \mathcal{C}_0^\psi : \mu_{\langle\psi\rangle}(|u| > t) < \infty, \forall t > 0\},$$

so that

$$\mathcal{C}_\sigma^\psi \subset \mathcal{C}_{0+}^\psi \subset \mathcal{C}_0^\psi, \qquad \forall \sigma \in \Sigma.$$

Of course, $\mathcal{C}_{0+}^\psi$ is also a contraction-stable vector space.

Here is the main result of the paper.

(3.10) Theorem. *Fix $\sigma \in \Sigma \cup \{0+\}$. The symmetric form $(\mathcal{E}^\psi, \mathcal{C}_\sigma^\psi)$ is closable, and its closure $(\mathcal{E}_\sigma^\psi, \mathcal{D}_\sigma^\psi)$ coincides with $(\mathcal{E}^Y, \mathcal{D}^Y)$.*

Proof. Fix $\sigma \in \Sigma \cup \{0+\}$. Let us begin by observing that the closability of $(\mathcal{E}^\psi, \mathcal{C}_\sigma^\psi)$ follows as in the proof of part (a) of (3.8) since $\mathcal{C}_\sigma^\psi \subset \mathcal{D}_{\rm ref}^Y$. Also, $\mathcal{D}^Y \subset \mathcal{D}_\sigma^\psi$ by virtue of (3.4)(b). Thus, to complete the proof we need only show that $\mathcal{C}_\sigma^\psi \subset \mathcal{D}^Y$, and for this it suffices to restrict attention to $\sigma \in \Sigma$. Define, for $t \ge 0$,

$$g(t) := \int_0^t [\sigma(s) \wedge 1]\, ds.$$

Notice that $\|g'\|_\infty \le 1$ and that $g(t)/t \le 1 \wedge \sqrt{\sigma(t)}$ for all $t \ge 0$.

Now fix $u \in p\mathcal{C}_\sigma^\psi$. Let $(f_n) \subset C_c^\infty(\mathbf{R})$ be chosen such that (i) $f_n = 1$ on $[-(n-1), n-1]$, (ii) $f_n = 0$ off $[-n, n]$, and (iii) $\|f_n'\|_\infty \le 2$. Define $u_n := (u - n^{-1})^+$ and

$$w_n := u \cdot g\circ u_n \cdot f_n\circ \log\psi, \qquad n = 1, 2, \ldots.$$

Clearly $w_n \in \mathcal{D}^X \cap L^2(\mu)$ and w_n vanishes off the set

$$G_n := \{u > 1/n\} \cap \{|\log\psi| < n\}$$

which differs from a finely open set by at most an (X, m)-polar subset of E_ψ. Moreover,

$$\mu_{\langle\psi\rangle}(G_n) \le \mu_{\langle\psi\rangle}(u > 1/n) = \mu_{\langle\psi\rangle}(\sigma\circ u \ge \sigma(1/n)) \le \mu_{\langle\psi\rangle}(\sigma\circ u)/\sigma(1/n) < \infty.$$

It now follows from (3.4)(a) that $w_n \in \mathcal{D}^Y$. By (2.8), since $g(t) \le t$,

$$\begin{aligned}
\mu_{\langle w_n\rangle} &\le 4[\, u^2 (g\circ u_n)^2 (f_n'\circ \log\psi)^2 \psi^{-2} \mu_{\langle\psi\rangle} \\
&\qquad\qquad + u^2 (f_n\circ\log\psi)^2 (g'\circ u_n)^2 \mu_{\langle u_n\rangle} + (g\circ u_n)^2 (f_n\circ\log\psi)^2 \mu_{\langle u\rangle}\,] \\
&\le 4\left[u^2 (g\circ u)^2 \psi^{-2} \mu_{\langle\psi\rangle} + \|u\|_\infty^2 \mu_{\langle u_n\rangle} + \|u\|_\infty^2 \mu_{\langle u\rangle}\right] \\
&\le 4\left[u^2 (g\circ u)^2 \psi^{-2} \mu_{\langle\psi\rangle} + 2\|u\|_\infty^2 \mu_{\langle u\rangle}\right].
\end{aligned}$$

Recalling that $g(t)^2 \le t^2\sigma(t)$, we therefore have

$$\mathcal{E}^\psi(w_n) \le 4\|u\|_\infty^4 \mu_{\langle\psi\rangle}(\sigma\circ u) + 16\|u\|_\infty^2 \mathcal{E}^\psi(u) < \infty, \qquad \forall n \in \mathbf{N}.$$

Since $\mu(w_n^2) \le \|u\|_\infty^2 \mu(u^2) < \infty$, the Banach-Saks theorem implies that there is a subsequence of (w_n) whose Cesàro means converge strongly in $\mathcal{D}^Y$ to some $w \in \mathcal{D}^Y$. This limit must coincide with the (μ-a.e.) pointwise limit $u \cdot g\circ u = \lim_n w_n$. Therefore $u \cdot g\circ u$ is an element of $\mathcal{D}^Y$. Now u, being an element of $\mathcal{C}_\sigma^\psi$, lies in $\mathcal{D}_{\text{ref}}^Y$. As such, u admits a decomposition $u = u_0 + h$ where $u_0 \in bp\mathcal{D}_e^Y$ ($\mathcal{D}_e^Y$ is the extended Dirichlet space associated with Y) and h is is bounded, positive and harmonic with respect to Y. More precisely, h can be represented as

$$h(x) = \mathbf{Q}^x(\Psi), \qquad (Y,\mu)\text{-q.e. } x \in E_\psi, \tag{3.11}$$

where Ψ is a positive bounded random variable vanishing off $\{\zeta < \infty\}$, and $\Psi\circ\theta_t = \Psi$ for all $t \in [0, \zeta[$, $\mathbf{Q}^\mu$-a.s. See [**1**]. By [**4**; (5.7)] applied to Y, $\lim_{t\uparrow\zeta} u_0(Y_t) = 0$, $\mathbf{Q}^\mu$-a.s. since $u_0 \in \mathcal{D}_e^Y$. For the same reason, $\lim_{t\uparrow\zeta} w(Y_t) = 0$, $\mathbf{Q}^\mu$-a.s. on $\{\zeta < \infty\}$. This in turn implies that $\lim_{t\uparrow\zeta} u(Y_t) = 0$, $\mathbf{Q}^\mu$-a.s. on $\{\zeta < \infty\}$, because $w = u \cdot g\circ u$ and $g(t) > 0$ for $t > 0$. Thus, $\lim_{t\uparrow\zeta} h(Y_t) = 0$, $\mathbf{Q}^\mu$-a.s. Now h is bounded, so choosing a sequence (T_n) of Y-stopping times announcing ζ, we have, using (3.11) and the strong Markov property,

$$\begin{aligned}
h(x) &= \mathbf{Q}^x(\Psi) = \mathbf{Q}^x(\Psi; T_n < \zeta) \\
&= \mathbf{Q}^x(\Psi\circ\theta_{T_n}; T_n < \zeta) = \mathbf{Q}^x(h(Y_{T_n}); T_n < \zeta) \to 0,\ n \to \infty, \qquad Y\text{-q.e. } x \in E_\psi.
\end{aligned} \tag{3.12}$$

Thus, $h = 0$ and $u = u_0$, μ-a.e. Consequently, $u = u_0 \in \mathcal{D}_e^Y \cap L^2(\mu) = \mathcal{D}^Y$. ∎

In the following corollary of Theorem (3.10) we present three simple sufficient conditions under which $\mathcal{D}^Y = \mathcal{D}_0^\psi$.

(3.13) Corollary. *Fix $\psi \in p\mathcal{D}_{\rm loc}$. Then each of the following conditions implies that $\mathcal{D}^Y = \mathcal{D}_0^\psi$:*

(a) $\mu_{\langle\psi\rangle}(E) < \infty$.

(b) There exist $p \geq 1$ and a finite constant $C_p > 0$ such that $\left[\int_E |u|^p \, d\mu_{\langle\psi\rangle}\right]^{1/p} \leq C_p \cdot \sqrt{\mathcal{E}(u,u)}$ for all $u \in \mathcal{D}^X$.

(c) $\mu_{\langle\psi\rangle} U \leq C \cdot m$ for some finite constant $C > 0$.

Proof. The sufficiency of condition (a) follows from Theorem (3.10) with $\sigma(t) \equiv 1$, and that of (b) with $\sigma(t) = t^p$. The sufficiency of (c) follows from (b), because [**7**; (1.19)] tells us that if (c) holds then $\int_E u^2 \, d\mu_{\langle\psi\rangle} \leq C^2 \mathcal{E}(u,u)$ for all $u \in \mathcal{D}^X$. ∎

(3.14) Remark. Let Cap be the 1-capacity associated with $(\mathcal{E}, \mathcal{D})$. If $p \geq 2$, then condition (b) of Corollary (3.13) holds if and only if there is a finite constant $\widetilde{C}_p > 0$ such that

$$\mu_{\langle\psi\rangle}(K) \leq \widetilde{C}_p \mathrm{Cap}(K)^{p/2}, \qquad \forall \text{ compact } K \subset E. \tag{3.15}$$

See Theorem 3.1 of [**9**].

Our final result is an extension of the principal work (Theorem 1.5) of [**3**]. The special case $b \equiv 1$ recovers (3.13)(a).

(3.16) Theorem. *Suppose there exists $b \in p\mathcal{D}_{\rm loc}$ such that $\mu_{\langle b\rangle} \leq m$. If $\psi \in p\mathcal{D}_{\rm loc}$ satisfies $\mu_{\langle\psi\rangle}[b \leq n] < \infty$ for each $n \in \mathbf{N}$, then $\mathcal{D}^Y = \mathcal{D}_0^\psi$.*

Proof. The proof is similar to that of Theorem (3.10), though much simpler; as there it suffices to show that $\mathcal{C}_0^\psi \subset \mathcal{D}^Y$. Define $\rho_n := (n-b)^+ \wedge 1$ for $n \in \mathbf{N}$. Then $\rho_n \in bp\mathcal{D}_{\rm loc}$, and if $u \in p\mathcal{C}_0^\psi$ then as in the proof of (3.1) we have $w_n := u\rho_n \in \mathcal{D}^Y$ and $\sup_n[\mathcal{E}^\psi(w_n) + \mu(w_n^2)] < \infty$. The ($\mu$-a.e.) pointwise limit $u = \lim_n w_n$ is therefore an element of $\mathcal{D}^Y$. This proves that $\mathcal{C}_0^\psi \subset \mathcal{D}^Y$. ∎

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