Note on the Grüss Inequality

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be a bounded random variable with

\begin{equation}
(1) \quad m_X := \text{ess inf} \ X, \quad M_X := \text{ess sup} \ X.
\end{equation}

Proposition.

\begin{equation}
(2) \quad |\text{Cov}(X, Y)| \leq \frac{(M_X - m_X)(M_Y - m_Y)}{4}.
\end{equation}

Proof. The general case follows immediately from the special case $X = Y$ and the Cauchy-Schwarz inequality. Thus we shall assume that $X = Y$ and show that

\begin{equation}
(3) \quad \text{Var}(X) \leq \frac{(M_X - m_X)^2}{4}.
\end{equation}

In proving (3) we first assume that $X$ has the following additional symmetry properties:

\begin{equation}
(4) \quad E(X) = 0 \quad \text{and} \quad m_X = -M_X.
\end{equation}

Then

\begin{equation}
(5) \quad \text{Var}(X) = E(X^2) \leq M_X^2 = [(M_X - m_X)/2]^2 = \frac{(M_X - m_X)^2}{4}.
\end{equation}

In general, apply the preceding to $X^* := Z(X - c)$, where $c := (M_X + m_X)/2$ and $Z$ is a random variable independent of $X$ with $P[Z = 1] = P[Z = -1] = 1/2$. Clearly $M_{X^*} = -m_{X^*} = (M_X - m_X)/2$, and

\begin{equation}
(6) \quad \text{Var}(X) \leq E[(X - c)^2] = \text{Var}(X^*) \leq \frac{(M_{X^*} - m_{X^*})^2}{4} = \frac{(M_X - m_X)^2}{4}.
\end{equation}

\[ \square \]

It is evident from (5) and (6) that equality holds in (3) if and only if

\begin{equation}
(7) \quad P[X^* = M_{X^*}] = P[X^* = m_{X^*}] = \frac{1}{2},
\end{equation}

or what amounts to the same thing

\begin{equation}
(8) \quad P[X \in \{M_X, m_X\}] = 1.
\end{equation}

That is, equality holds in (3) if and only if the essential range of $X$ contains at most two points.

References