HERMITE MARTINGALES

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The Hermite polynomials h_n, $n \in \mathbf{N}$, defined by the Rodrigues formulae

$$h_n(x) := (-1)^n \exp(x^2/2) \frac{d^n}{dx^n} \exp(-x^2/2), \qquad x \in \mathbf{R}, \tag{1}$$

play an important role in the theory of Brownian motion; see, for example, [**3**], [**4**], [**6**]. In particular, if $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ is a filtered probability space on which is defined a standard one-dimensional Brownian motion $\{B_t; t \ge 0\}$ with $B_0 = 0$, then $\{t^{n/2} h_n(B_t/\sqrt{t}); t \ge 0\}$, is a martingale for every $n \in \mathbf{N}$.

An interesting converse, characterizing the Hermite polynomials, has recently been discovered by A. Plucińska [**5**]: If $n \ge 0$ is an integer, $h : \mathbf{R} \to \mathbf{R}$ is real analytic, and $t \mapsto t^{n/2} h(B_t/\sqrt{t})$ is a martingale, then h is proportional to h_n. Strictly speaking, this assertion is true only if we alter the initial state of the Brownian motion to ensure that $\mathbf{P}[B_0 = 0] < 1$. Indeed, for every real $p > 0$ there is a non-polynomial real analytic h such that $\{t^{p/2} h(B_t/\sqrt{t}); t \ge 0\}$ is a martingale, provided the Brownian motion satisfies $\mathbf{P}[B_0 = 0] = 1$; see part (b) of Theorem 1 below. Our purpose in this note is to give a new proof of (an extension of) Plucińska's Theorem.

As preparation we collect some known results concerning the connection between space-time harmonic functions and martingale functions of space-time Brownian motion. Let

$$p_t(x, y) := [2\pi t]^{-1/2} \exp(-(y - x)^2/2t)$$

denote the Brownian transition kernel, and define the corresponding semigroup of transition operators by

$$\begin{aligned} P_t f(x) : &= \int_{\mathbf{R}} p_t(x, y) f(y)\, dy \\ &= \mathbf{P}^x[f(B_t)] = \mathbf{P}[f(x + B_t)], \qquad x \in \mathbf{R}, t \ge 0. \end{aligned} \tag{2}$$

Here $\mathbf{P}^x$ denotes both the law of Brownian motion started at x and the associated expectation operator.

Lemma 1. *If $H : \mathbf{R} \times (0,\infty) \to \mathbf{R}$ is Borel measurable, then the following statements are equivalent:*

(a) $P_{t-s}[H(\cdot,t)](x) = H(x,s)$ for all $x \in \mathbf{R}$ and all $0 < s < t$;

(b) $P_{t-s}[H(\cdot,t)](x) = H(x,s)$ for Lebesgue a.e. $x \in \mathbf{R}$, for all $0 < s < t$, and $\mathbf{P}^x|H(B_t, t+r)| < \infty$ for all $x \in \mathbf{R}$ and all $r, t > 0$;

(c) $t \mapsto H(B_t, t+r)$ is a $\mathbf{P}^x$ martingale, for all $x \in \mathbf{R}$ and all $r > 0$.

Proof. The implication (a)⇒(b) is trivial, and (b)⇒(c) follows easily because the $\mathbf{P}^x$-distribution of B_s is absolutely continuous with respect to Lebesgue measure for all $x \in \mathbf{R}$ and all $s > 0$:

$$\mathbf{P}^x[H(B_t, t+r)|\mathcal{F}_s] = P_{t-s}[H(\cdot, t+r)](B_s) = H(B_s, s+r), \qquad \mathbf{P}^x\text{-a.s.}$$

Finally, if (c) holds then for $x \in \mathbf{R}$ and $r, t > 0$,

$$H(x,r) = \mathbf{P}^x[H(B_0, 0+r)] = \mathbf{P}^x[H(B_t, t+r)] = P_t[H(\cdot, t+r)](x),$$

which yields (a) after a change of variables. $\square$

Lemma 2. *Let $H : \mathbf{R} \times (0,\infty) \to \mathbf{R}$ be a function of class $C^{2,1}$.*

(i) The process $t \mapsto H(B_t, t+r)$ is a $\mathbf{P}^x$ local martingale for all $(x,r) \in \mathbf{R} \times (0,\infty)$ if and only if $\partial H/\partial t + \frac{1}{2}\partial^2 H/\partial x^2 \equiv 0$.

(ii) Suppose that $\partial H/\partial t + \frac{1}{2}\partial^2 H/\partial x^2 \equiv 0$ and that for each $T > 0$ there is a constant C_T such that $|H(x,t)| \le C_T \exp(x^2/2t)$ for all $(x,t) \in \mathbf{R} \times (0,T]$. Then $t \mapsto H(B_t, t+r)$ is a $\mathbf{P}^x$ martingale for all $x \in \mathbf{R}$ and all $r > 0$.

Proof. Assertion (i) follows immediately from Itô's formula. Assertion (ii) is a consequence of classical theorems on the well-posedness of the Cauchy problem. Let us fix $T > 0$ and $r > 0$, and define

$$K(x,t) := P_{T-t}[H(\cdot, T+r)](x), \qquad (x,t) \in \mathbf{R} \times [0,T].$$

Then K is a $C^{2,1}$ solution of $\partial H/\partial t + \frac{1}{2}\partial^2 H/\partial x^2 \equiv 0$ on $\mathbf{R} \times [0,T)$ with $K(x,T) = H(x, T+r)$ for all $x \in \mathbf{R}$, and

$$|K(x,t)| \le C \exp(k \cdot x^2), \qquad (x,t) \in \mathbf{R} \times [0,T],$$

for some constant $k > 0$; see Theorem 12 in Chapter 1 of [**2**]. By Theorem 16 *loc. cit.*, $K(x,t) = H(x, t+r)$ for all $(x,t) \in \mathbf{R} \times [0,T]$. That is

$$P_{T-t}[H(\cdot, T+r)](x) = H(x, t+r)$$

for all $(x,t) \in \mathbf{R} \times [0,T]$. Since $T > 0$ and $r > 0$ were arbitrary, part (ii) follows from Lemma 1. $\square$

Here is the main result of this note. One could relax the conditions imposed on α and h in part (a) (measurability and local boundedness would suffice); we leave this extension to the reader.

Theorem 1. *(a) Let* $h : \mathbf{R} \to \mathbf{R}$ *be of class* C^2*, and let* α *and* β *be* C^1 *mappings of* $(0, \infty)$ *into itself such that*

$$\alpha(1) = \beta(1) = 1 \qquad \textit{and} \qquad \beta(0+) = 0. \tag{3}$$

Define

$$H(x,t) := \alpha(t) \cdot h(x/\beta(t)), \qquad t > 0, x \in \mathbf{R}, \tag{4}$$

and suppose that

$$t \mapsto H(B_t, t+r) \textit{ is a } \mathbf{P}^x \textit{ local martingale, for all } x \in \mathbf{R} \textit{ and all } r > 0. \tag{5}$$

Then one of the following statements is true:

(i) h *is constant and* $\alpha \equiv 1$*.*

(ii) $h(x) = \text{Const.} \cdot x$ *and* $\alpha \equiv \beta$*.*

(iii) $\beta(t) = \sqrt{t}$ *for* $t > 0$*, there is a real number* p *such that* $\alpha(t) = t^{p/2}$ *for* $t > 0$*, and* h *satisfies the Hermite equation*

$$h''(x) - x \cdot h'(x) + p \cdot h(x) = 0, \qquad \forall x \in \mathbf{R}. \tag{6}$$

(b) Conversely, if h *is a* C^2 *function satisfying (6), then* $t \mapsto H(B_t, t+r)$ *is a* $\mathbf{P}^x$ *martingale for every* $x \in \mathbf{R}$ *and every* $r > 0$*, where* $H(x,t) := t^{p/2}h(x/\sqrt{t})$*. If, in addition,* $p > 0$*, then* $t \mapsto H(B_t, t)$ *is a* $\mathbf{P}^0$ *martingale.*

(c) If h *is a* C^2 *function such that* $t \mapsto t^{p/2}h(B_t/\sqrt{t})$ *is a* $\mathbf{P}^x$ *martingale for some* $x \neq 0$*, then* p *is a non-negative integer and* h *is proportional to the Hermite polynomial* h_p*.*

Proof. (a) By Lemma 2(i), H satisfies the (dual) heat equation $\partial H/\partial t + \frac{1}{2}\partial^2 H/\partial x^2 \equiv 0$; consequently,

$$\frac{1}{2}h''(x) - \beta(t)\beta'(t)xh'(x) + [\beta(t)]^2\frac{\alpha'(t)}{\alpha(t)}h(x) = 0, \qquad \forall t > 0, x \in \mathbf{R}. \tag{7}$$

If $\beta\beta'$ is non-constant then there are times $s, t > 0$ such that $c := \beta(t)\beta'(t) - \beta(s)\beta'(s)$ is non-zero. Fix such times and define $b := [\beta(t)]^2\frac{\alpha'(t)}{\alpha(t)} - [\beta(s)]^2\frac{\alpha'(s)}{\alpha(s)}$; then (7) implies

$$c \cdot xh'(x) = b \cdot h(x), \qquad \forall x \in \mathbf{R}. \tag{8}$$

Any solution of (8) must be of the form $h(x) = \text{Const.} \cdot x^\gamma$ for $x > 0$, where $\gamma := b/c$. For an h of this form to satisfy (7) (for $x > 0$) we must have $\gamma = 0$ or $\gamma = 1$. If $\gamma = 0$ then the C^2 solutions of (8) are constant; this is case (i) of part (a) of Theorem 1. If $\gamma = 1$ then $h(x) = \text{Const.} \cdot x$, which is case (ii).

Thus, with the exception of the trivial cases (i) and (ii), $\beta(t)\beta'(t)$ is constant, which means that $\beta(t) = \sqrt{t}$ for $t \ge 0$, because of (3). Inserting this expression for β into (7) we arrive at

$$h''(x) - xh'(x) + 2t\frac{\alpha'(t)}{\alpha(t)}h(x) = 0. \tag{9}$$

Unless h is identically 0 (which case has already been dealt with), (9) implies that $t \mapsto t\alpha'(t)/\alpha(t)$ is constant. In this case $\alpha(t) = t^{p/2}$ for some $p \in \mathbf{R}$, and (9) simplifies to (6).

(b) Fix $p \in \mathbf{R}$, let h solve (6), and define $H(x,t) := t^{p/2}h(x/\sqrt{t})$. The function h, being a solution of (6), can be expressed as $c_1Y_1(x) + c_2Y_2(x)$, where

$$Y_1(x) := M(-\tfrac{1}{2}p, \tfrac{1}{2}, \tfrac{1}{2}x^2), \quad Y_2(x) := xM(-\tfrac{1}{2}(p-1), \tfrac{3}{2}, \tfrac{1}{2}x^2) \tag{10}$$

are linearly independent solutions of (6); here $z \mapsto M(a,b,z)$ is the solution of Kummer's equation

$$zw''(z) + (b-z)w'(z) - aw(z) = 0$$

given by

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)}\frac{z^n}{n!}. \tag{11}$$

See 13.1.1, 13.1.2, 19.2.1 and 19.2.3 in [**1**]. For $b > 0$ as in the present situation, $M(a,b,z)$ is an entire function of z. Moreover, Y_1 (resp. Y_2) is a polynomial if and only if p is an even (resp. odd) non-negative integer. The asymptotic behavior of M is known [**1**; 13.1.4], and yields the estimate

$$|h(x)| \le \text{Const.} \cdot \exp(x^2/2) \cdot \Big[1 + |x|\Big]^{-p-1}. \tag{12}$$

Clearly (12) implies the bound appearing in part (ii) of Lemma 2. Moreover, because h satisfies (6), H satisfies $\partial H/\partial t + \frac{1}{2}\partial^2 H/\partial x^2 \equiv 0$. The first assertion therefore follows from Lemma 2(ii). Turning to the second assertion, if $p > 0$, then $\mathbf{P}^0|H(B_t,t)| < \infty$ by (12). The family $\{H(B_t,t); t > 0\}$ of $\mathbf{P}^0$-integrable random variables is a martingale because of Lemma 2(ii). By the backward martingale convergence theorem, the limit $\lim_{t\downarrow 0} H(B_t,t)$ exists $\mathbf{P}^0$-a.s. and in $L^1(\mathbf{P}^0)$; the $\mathbf{P}^0$-a.s. limit is easily seen to be 0, by (12) and the law of the iterated logarithm. Consequently, if $H(B_0,0)$ is understood to be 0, then $\{H(B_t,t); t \ge 0\}$ is a $\mathbf{P}^0$ martingale.

(c) Let h be a C^2 function such that $t \mapsto t^{p/2}h(B_t/\sqrt{t})$ is a $\mathbf{P}^x$ martingale for some $x \neq 0$. Then h satisfies (6), and unless h is a polynomial the estimate (12) can be strengthened to an asymptotic equivalence:

$$|h(x)| \sim \text{Const.} \cdot \exp(x^2/2) \cdot |x|^{-p-1}, \qquad |x| \to \infty.$$

See 13.1.4 in [**1**]. The $\mathbf{P}^x$ integrability of $h(B_t/\sqrt{t})$, for $t = 1$, implies that for N sufficiently large

$$\begin{aligned} \infty &> \int_{\mathbf{R}} |h(y)| \exp(-(y-x)^2/2)\, dy \\ &\geq \text{Const.} \cdot \exp(-x^2/2) \int_{|y| \geq N} \exp(xy)|y|^{-p-1}\, dy, \end{aligned}$$

which is clearly absurd because $x \neq 0$. Thus, h must be a polynomial. In view of (10) and (11), the only polynomial solutions of (6) occur when p is a non-negative integer, and any such polynomial solution is proportional to h_p. ☐

Remark. Only the local martingale property of $t^{p/2}h(B_t/\sqrt{t})$ and the integrability of $h(B_1)$ were used in the proof of (c). An alternative proof, which uses more fully the hypothesis that $t^{p/2}h(B_t/\sqrt{t})$ is a martingale, was suggested by the referee: If $t^{p/2}h(B_t/\sqrt{t})$ is a $\mathbf{P}^x$ martingale for some $x \neq 0$, then $\lim_{t\downarrow 0} t^{p/2}h(B_t/\sqrt{t})$ exists $\mathbf{P}^x$ almost surely. This implies the existence of $\lim_{t\downarrow 0} t^{p/2}h(x/\sqrt{t})$, which forces the (entire!) function h to have a pole (of order at most p) at infinity. In other words, h must be a polynomial.

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