Note on Kemeny’s Constant

by

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In a recent Letter to the Editor [1] of the Journal of Applied Probability, O. Angel and M. Holmes have proved a conjecture found in [2] that Kemeny’s constant is infinite in the case of a discrete time Markov chain with countably infinite state space. We give here a different proof, showing how to reduce the problem to the finite state space case, where Hunter [3] has provided a lower bound.

Let \( X = (X_k)_{k \geq 0} \) be an positive recurrent Markov chain with countably infinite state space \( E \). As such \( X \) admits a unique stationary distribution \( \pi \). As usual, \( T_y(X) := \inf\{k \geq 1 : X_k = y\} \) denotes the hitting time of state \( y \in E \). It is well known that when \( E \) is finite the expected hitting time

\[
\sum_{y \in E} E^x[T_y(X)] \pi(y),
\]

of a state chosen independently of \( X \) using \( \pi \), is a constant not depending on the starting state \( x \), the so-called Kemeny constant. For discussion and further references see [2] and [3]. Hunter in [3] has proved the lower bound

\[
\sum_{y \in E} E^x[T_y(X)] \pi(y) \geq (\text{card}(E) + 1)/2, \quad x \in E,
\]

but, as remarked in [2], was unable to extend the argument to the infinite state space case. The authors of [1] proved a local version of Hunter’s lower bound, valid whether the state space is finite or infinite, and used that to prove the conjecture of [2] mentioned in the first paragraph.

Let \( E_1 \subset E_2 \subset \cdots \) be an increasing sequence of finite subsets of \( E \) with union all of \( E \), such that \( E_n \) has cardinality \( n \). Fix \( n \) and consider the Markov chain \( Y^{(n)} \) obtained by observing \( X \) only when it is in \( E_n \). More precisely, for \( X_0 = x \in E_n \) define \( T_0^{(n)} := 0 \), and \( T_{k+1}^{(n)} := \min\{j > T_k^{(n)} : X_j \in E_n\}, k = 0, 1, 2, \ldots \). Then \( Y_k^{(n)} := X_{T_k^{(n)}} \), \( k = 0, 1, 2, \ldots \) defines a positive recurrent Markov chain on \( E_n \).

By the ergodic theorem, if \( A \subset E_n \), then

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^n 1_A(Y_k^{(n)})}{m} = \frac{\sum_{k=1}^{T_m^{(n)}} 1_A(X_k)}{\sum_{j=1}^{T_m^{(n)}} 1_{E_n}(X_j)} = \frac{\pi(A)}{\pi(E_n)},
\]

\( \mathbb{P}^x \)-a.s. for \( x \in E_n \). It follows that \( [\pi(E_n)]^{-1} \pi \) is the stationary distribution for \( Y^{(n)} \).

Evidently

\[
T_y(Y^{(n)}) \leq T_y(X)
\]

for all \( y \in E_n \). Consequently,

\[
\sum_{y \in E_n} E^x[T_x(Y^{(n)})] \pi(y) \leq \sum_{y \in E_n} E^x[T_y(X)] \pi(y) \leq \sum_{y \in E} E^x[T_y(X)] \pi(y), \quad \forall x \in E_n,
\]
and so by (2) applied to $Y^{(n)}$

$$\sum_{y \in E} E^x[T_y(X)] \pi(y) \geq \pi(E_n) \cdot (n + 1)/2, \quad \forall x \in E_n.$$ 

Sending $n$ off to infinity we find that

$$\sum_{y \in E} E^x[T_y(X)] \pi(y) = \infty, \quad \forall x \in E,$$

as desired.

References

