Note on a paper of Chen & Tweedie

In a recent paper [**1**], P.-D. Chen and R.L. Tweedie make use of the natural embedding of a separable and separated measurable space in the compact cube $\{0,1\}^{\mathbb{N}}$. The point of this note is to show that a well-known representation of functions measurable with respect to the σ-algebra generated by a sequence of functions can be substituted for the rather intricate arguments found in [**1**].

Let $(X, \mathcal{X})$ be a measurable space such that $\mathcal{X}$ is countably generated and separated. The latter term means that if x and y are distinct points of X then there exists $A \in \mathcal{X}$ such that $x \in A$ and $y \in A^c$.

Let I denote the cube $\{0,1\}^{\mathbb{N}}$ endowed with its natural (compact, metrizable) topology, and let $\mathcal{I}$ denote the class of Borel subsets of I. Let $\{A_1, A_2, \ldots\}$ be a sequence of elements of $\mathcal{X}$ such that $\sigma\{A_n : n \in \mathbf{N}\} = \mathcal{X}$. Define a mapping $T : X \to I$ by the formula $T(x) = (1_{A_1}(x), 1_{A_2}(x), \ldots)$. Clearly T is injective and $\mathcal{X}/\mathcal{I}$-measurable. If we endow the image $Y := T(X)$ with the trace σ-algebra $\mathcal{Y} := \mathcal{I} \cap Y$, then T is an isomorphism of the measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$. In general, $Y \notin \mathcal{I}$; indeed, $Y \in \mathcal{I}$ if and only if $(X, \mathcal{X})$ is a measurable Lusin space (*i.e.*, isomorphic to a Borel subspace of some compact metrizable space).

Put $Z := \overline{Y}$. Then Z is compact, and the trace σ-algebra $\mathcal{Z} := \mathcal{I} \cap Z$ coincides with the class of Borel subsets of Z. Of course, $\mathcal{Y} = \mathcal{Z} \cap Y$.

Lemma. *Let $(F, \mathcal{F})$ be a Lusin space and let $f : X \to F$ be $\mathcal{X}/\mathcal{F}$-measurable. Then there exists an $\mathcal{Z}/\mathcal{F}$-measurable function $\hat{f} : Z \to F$ such that $f = \hat{f} \circ T$.*

Proof. The assertion is trivial if F is countable. If F is uncountable, then it is Borel isomorphic to the unit interval $[0,1]$. Since both the hypotheses and conclusions are isomorphism invariant we can assume without loss of generality that $F = [0,1]$. But in this case the assertion is a standard consequence of the monotone class theorem. □

If P is a probability measure on $(X, \mathcal{X})$, let $Q := T_* P$ denote the image of P under T, viewed as a measure on $(Z, \mathcal{Z})$:

$$Q(H) := P(T^{-1}H), \qquad \forall H \in \mathcal{Z}.$$

The class of images $\mathcal{Q} := \{T_* P : P \in \mathcal{P}(E)\}$ does not exhaust $\mathcal{P}(Z)$.* In fact, $Q \in \mathcal{P}(Z)$ lies in $\mathcal{Q}$ if and only if the Q outer measure of Y is 1.

Say that $A \in \mathcal{Z}$ is $\mathcal{Q}$-*negligible* provided $Q(A) = 0$ for all $Q \in \mathcal{Q}$. Likewise, $\mathcal{Z}$-measurable functions f and g are $\mathcal{Q}$-*equivalent* ($f \sim g$) provided $\{f \neq g\}$ is $\mathcal{Q}$-negligible. Clearly $f \sim g$ if and only if $\{f \neq g\} \subset Z \setminus Y$. In particular, the function $\hat{f}$ provided by the Lemma is uniquely determined modulo $\mathcal{Q}$-equivalence.

* $\mathcal{P}(Z)$ denotes the class of probability measures on $(Z, \mathcal{Z})$, etc.

Let us now see how the Lemma provides an easy way to "lift" a Markov kernel on $(X, \mathcal{X})$ to a Markov kernel on $(Z, \mathcal{Z})$. By the term "Markov kernel" (on $(X, \mathcal{X})$, say) we mean a mapping $(x, A) \mapsto K(x, A)$ from $X \times \mathcal{X}$ to $[0, 1]$ such that $x \mapsto K(x, A)$ is $\mathcal{X}$-measurable for each $A \in \mathcal{A}$ and $A \mapsto K(X, A)$ is an element of $\mathcal{P}(X)$ for all $x \in X$. Here is the main result of section 3 of [**1**]:

Theorem. *Let K be a Markov kernel on $(X, \mathcal{X})$. Then there exists a Markov kernel $\hat{K}$ on $(Z, \mathcal{Z})$ such that $K(x, T^{-1}H) = \hat{K}(T(x), H)$ for all $x \in X$ and all $H \in \mathcal{Z}$. If $\bar{K}$ is another such "lifting" of K, then $\{z \in Z : \bar{K}(z, \cdot) \neq \hat{K}(z, \cdot)\}$ is $\mathcal{Q}$-negligible.*

Proof. For each fixed $x \in X$, the mapping $\mathcal{Z} \ni H \mapsto K(x, T^{-1}H)$ defines an element $K_*(x, \cdot)$ of $\mathcal{P}(Z)$ (in fact, an element of $\mathcal{Q}$). Since Z is compact metrizable, so is $\mathcal{P}(Z)$. By the Lemma (applied to the function $f : x \mapsto K_*(x, \cdot)$, which is a measurable mapping of X into $\mathcal{P}(Z)$) there is a measurable function $\hat{f} : Z \to \mathcal{P}(Z)$ such that $f = \hat{f} \circ T$. Evidently, $\hat{f}$ determines a Markov kernel $\hat{K}$ on $(Z, \mathcal{Z})$ by the formula $\hat{K}(z, H) = [\hat{f}(z)](H)$. In particular, $\hat{K}(T(x), H) = [\hat{f}(T(x))](H) = [f(x)](H) = K(x, T^{-1}H)$, for all $H \in \mathcal{Z}$ and $x \in X$. The uniqueness assertion is obvious. $\square$

References

[1] P.-D. Chen and R.L. Tweedie: Orthogonal measures and absorbing sets for Markov chains, *Math. Proc. Camb. Phil. Soc.* **121** (1997) 101–113).