Lévy Systems and Time Changes

by

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ABSTRACT

The Lévy system for a Markov process X provides a convenient description of the distribution of the totally inaccessible jumps of the process. We examine the effect of time change (by the inverse of a not necessarily strictly increasing CAF A) on the Lévy system, in a general context. Our basic hypothesis (beyond the "right" Markov property) is that the "irregular" exits from the fine support of A occur at totally inaccessible times. This condition permits the construction of a predictable exit system (à la Maisonneuve), the key tool for our time change theorem.

The second part of the paper is devoted to some implications of the preceding in a (weak, moderate Markov) duality setting. Fixing an excessive measure m (to serve as duality measure) we obtain formulas relating the "killing" and "jump" measures for the time-changed process to the analogous objects for the original process. These formulas extend, to a very general context, recent work of Chen, Fukushima, and Ying. The key to our development is the Kuznetsov process associated with X and m, and the associated moderate Markov dual process $\widehat{X}$. Using $\widehat{X}$ and some excursion theory, we exhibit a general method for construction excessive measures for X from excessive measures for the time-changed process.

Key words and phrases: Lévy system, exit system, time change, Markov process, continuous additive functional, excessive measure, Kuznetsov process.

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1. Introduction.

Let $X = (X_t, \mathbf{P}^x)$ be a right Markov process with state space E. The *Lévy system* of X describes the intensity with which X makes totally inaccessible jumps of specified types. It consists of a continuous additive functional H and a kernel N on $(E, \mathcal{E})$ such that

$$t \mapsto \sum_{s \le t} \Phi(X^r_{s-}, X_s) - \int_0^t N(X_s, \Phi)\, dH_s \tag{1.1}$$

is a $\mathbf{P}^x$-martingale for each $x \in E$, provided $\mathbf{P}^x \int_0^t N(X_s, |\Phi|)\, dH_s < \infty$ for each $t > 0$. In (1.1), X^r_{s-} is the left limit of X at time $s > 0$ taken in a suitable Ray topology on E, Φ is a product measurable function on $E \times E$ with $\Phi(x, x) = 0$ for all $x \in E$, and $N(x, \Phi) := \int_E N(x, dy)\Phi(x, y)$. Intuitively, the rate at which jumps from $x \in E$ to $\Lambda \in \mathcal{E}$ occur, *relative to the clock* H, is $N(x, \Lambda)$. The notion of Lévy system, which is a far-reaching generalization of the Itô-Lévy description of the jumps of a Lévy process, is due to S. Watanabe [**W64**]. He constructed Lévy systems for Hunt processes satisfying Meyer's hypothesis (L) (= the existence of a reference measure). Lévy systems for general right (and Ray) processes, without (L), were constructed by Benveniste and Jacod in [**BJ73**].

Suppose that in addition to the right process X we have a CAF A with right continuous inverse τ. Let F denote the fine support of A; thus A increases when (and only when) X is in F. It is well known that the time-changed process $\widetilde{X}_t := X_{\tau(t)}$, $t \ge 0$, is a right process with state space F. Our goal in this paper is to express the Lévy system $(\widetilde{N}, \widetilde{H})$ of $\widetilde{X}$ in terms of (N, H) and the (predictable) *exit system* $(\mathbf{P}^{\bullet}_{\mathrm{pr}}, C)$; the latter describes the ways in which X exits the fine support F. The need for this second ingredient stems from the fact that A need not be strictly increasing—some of the totally inaccessible jumps of $\widetilde{X}$ correspond to totally inaccessible jumps of X, while others are generated by the excursions of X from F. The first work in this direction of which we are aware is that of H. Gzyl [**Gz77**]. The recent work of Chen, Fukushima, and Ying [**CFY06a, CFY06b**] has been the direct inspiration for the present study. The effect of time change on symmetric Markov processes is considered in [**CFY06a**] (see also [**LJ77**] for the case of "nearly symmetric" Hunt processes, and [**FHY04**] for symmetric *diffusions*). The same issues are examined in [**CFY06b**] for a standard process in weak duality with a second standard process, under the condition that semipolar sets are m-polar (m being the duality measure). In trying to understand [**CFY06b**], we came to realize that neither duality nor a restriction on semipolar sets was crucial for the discussion. Rather, the key seemed to be that the "irregular" exits of X from F occur at totally inaccessible times, and we have taken this simple probabilistic hypothesis to be the basis for our study.

In section 2 we describe the hypotheses that will be in force throughout the paper, and we recall the basic facts about exit systems and Lévy systems. In section 3, following Maisonneuve [**Ma75**], we define the concept of *predictable* exit system and, in Theorem (3.12), provide several conditions equivalent to the existence of a predictable exit system. In a short section 4 we recall the definition of the time-changed process $\widetilde{X}$ induced by a continuous additive functional A of the basic process X. Section 5 contains one of the main results of the paper. Namely, the Lévy system of $\widetilde{X}$ is expressed in terms of the Lévy system of X and the predictable exit system describing the excursions of X away from the fine support F of A; see Theorem (5.10). In section 6 we assume the existence of an excessive measure m, and recall several constructs depending on m: In particular, the Kuznetsov measure $\mathbf{Q}_m$ and associated processes Y and Y^*. We introduce the "jump measure" $\mathcal{J}$ and the "killing measure" $\mathcal{K}$ associated with X and m, and we express them in terms of the Lévy system of X. Following [**CFY06b**] we then define the Feller measure Λ and the supplementary Feller measure δ associated with excursions from F. Formulas (6.25) and (6.26) relate Λ and δ explicitly to the predictable exit system. The main result of this section, Theorem (6.31), gives formulas for the jump measure $\widetilde{\mathcal{J}}$ and the killing measure $\widetilde{\mathcal{K}}$ of $\widetilde{X}$ in terms of $\mathcal{J}$, $\mathcal{K}$, Λ, and δ. Section 7 introduces the left-continuous moderate markov process $\widehat{X}$ in weak duality with X relative to m. Using this duality we extend some of the results about excursions from a regular point presented in [**FG06**] to excursions from a finely perfect nearly Borel set F for which a predictable exit system exists.

We close this introduction with a few words on notation. If $(F, \mathcal{F}, \mu)$ is a measure space, then $b\mathcal{F}$ (resp. $p\mathcal{F}$) denotes the class of bounded real-valued (resp. $[0,\infty]$-valued) $\mathcal{F}$-measurable functions on F. For $f \in p\mathcal{F}$ we may use $\mu(f)$ or $\langle \mu, f\rangle$ to denote the integral $\int_F f\,d\mu$; similarly, if $D \in \mathcal{F}$ then $\mu(f; D)$ denotes $\int_D f\,d\mu$. On the other hand $f\mu$ denotes the measure $f(x)\mu(dx)$ and $\mu|_D$ the restriction of μ to D. We write $\mathcal{F}^*$ for the universal completion of $\mathcal{F}$; that is $\mathcal{F}^* = \cap_\nu \mathcal{F}^\nu$, where $\mathcal{F}^\nu$ is the ν-completion of $\mathcal{F}$ and the intersection runs over all finite measures on $(F, \mathcal{F})$. If $(E, \mathcal{E})$ is a second measurable space and $K = K(x, dy)$ is a kernel from $(F, \mathcal{F})$ to $(E, \mathcal{E})$ (i.e., $F \ni x \mapsto K(x, A)$ is $\mathcal{F}$-measurable for each $A \in \mathcal{E}$ and $K(x, \cdot)$ is a measure on $(E, \mathcal{E})$ for each $x \in F$), then we write μK for the measure $A \mapsto \int_F \mu(dx)K(x, A)$ and Kf for the function $x \mapsto \int_E K(x, dy)f(y)$. We shall use $\mathcal{B}$ to denote the Borel subsets of the real line $\mathbf{R}$. If T is a stopping time, then $[\![T]\!]$ denotes the graph $\{(\omega, t) \in \Omega \times [0,\infty[: t = T(\omega)\}$.

2. Preliminaries.

Throughout the paper $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, \mathbf{P}^x)$ will denote the canonical realization of a Borel right Markov process with state space $(E,\mathcal{E})$. We shall use the standard notation for Markov processes as found, for example, in [**BG68**], [**G90**], [**DM87**] and [**Sh88**]. In short, X is a strong Markov process with right continuous sample paths, the state space E (with Borel σ-field $\mathcal{E}$) is homeomorphic to a Borel subset of a compact metric space, and the transition semigroup $(P_t)_{t\ge 0}$ of X preserves the class $b\mathcal{E}$ of bounded $\mathcal{E}$-measurable functions. It follows that the resolvent operators $U^q := \int_0^\infty e^{-qt}P_t\,dt$, $q\ge 0$, also preserve Borel measurability. We shall write U for U^0. We allow the transition semigroup (P_t) to be subMarkovian: $P_t1(x)\le 1$ for all $x\in E$ and all $t\ge 0$. To allow for the possibility $P_t1_E(x)<1$, an absorbing cemetery state Δ is adjoined to E as an isolated point, and the process is sent to Δ at its lifetime ζ. Thus X takes values in $E_\Delta := E\cup\{\Delta\}$ (endowed with the σ-field $\mathcal{E}_\Delta := \mathcal{E}\vee\{\Delta\}$; until section 6, the cemetery state will play no special role.

We write $\mathcal{E}^e$ for the σ-algebra on E generated by the 1-excessive functions. Because the semigroup (P_t) is Borel, all 1-excessive functions are nearly Borel measurable; consequently, $\mathcal{E}^e$ is contained in the σ-algebra of nearly Borel sets.

One of our concerns will be the excursions induced by a CAF A; to this end we recall Maisonneuve's notion of optional exit system. The related notion of predictable exit system will be discussed in section 3. It is known that the stopping time $\tau(0)=\inf\{t : A_t>0\}$ is equal a.s. to the hitting time $T_F := \inf\{t>0 : X_t\in F\}$ of the *fine support* F of A, defined by

$$F := \{x\in E : \mathbf{P}^x[\tau(0)=0]=1\}. \tag{2.1}$$

The set F is $\mathcal{E}^e$-measurable and *finely perfect* in the sense that $F=F^r$, where $F^r := \{x\in E : \mathbf{P}^x[T_F=0]=1\}$ denotes the set of points regular for F. Consequently, the optional set $\{X\in F\} := \{(\omega,t) : X_t(\omega)\in F\}$ has ω-sections that are right closed and without isolated points, almost surely. Let M be the (optional) subset of $\Omega\times[0,\infty[$ with ω-section $M(\omega)$ equal to the closure in $[0,\infty[$ of the visiting set $\{t\ge 0 : X_t(\omega)\in F\}$, for each $\omega\in\Omega$. The complement of $M(\omega)$ comprises a countable union of disjoint open intervals. We write $G(\omega)$ for the collection of strictly positive left endpoints of these "contiguous intervals". The associated random set G is progressively measurable, but not in general optional. More precisely, the "regular" part of G, given by

$$G^r := \{(\omega,s)\in G : X_s(\omega)\in F\}, \tag{2.2}$$

has evanescent intersection with the graph of any stopping time, while the "irregular" part

$$G^i := \{(\omega,s)\in G : X_s(\omega)\notin F\} \tag{2.3}$$

is a countable union of graphs of stopping times.

According to Maisonneuve [**Ma75**] there is an *optional exit system* consisting of an AF B with bounded 1-potential, and a kernel $\mathbf{P}^{\bullet}_{\text{op}}$ from $(E_\Delta, \mathcal{E}^*_\Delta)$ to $(\Omega, \mathcal{F}^*)$ such that

$$\mathbf{P}^x \sum_{s \in G} Z_s \, \Phi \circ \theta_s = \mathbf{P}^x \int_0^\infty Z_s \, \mathbf{P}^{X_s}_{\text{op}}[\Phi] \, dB_s, \tag{2.4}$$

for all optional $Z \geq 0$, $\Phi \in p\mathcal{F}^*$, and $x \in E_\Delta$. ($\mathbf{P}^{\Delta}_{\text{op}}$ is the point mass at the dead path $[\Delta]$.) We can (and do) take the continuous part B^c of B to be the dual predictable projection of the raw AF

$$t \mapsto \sum_{s \leq t, s \in G^r} [1 - \exp(-T_F)] \circ \theta_s + \int_0^t 1_F(X_s) \, ds,$$

and the discontinuous part of B to be

$$B^d_t := \sum_{s \leq t, s \in G^i} \mathbf{P}^{X_s}[1 - \exp(-T_F)]. \tag{2.5}$$

Notice that B^c grows only when X is in F. In view of Motoo's theorem [**Sh88**; (66.2)], there exists $\ell \in p\mathcal{E}^e$ such that

$$\int_0^t 1_F(X_s) \, ds = \int_0^t \ell(X_s) \, dB^c_s = \int_0^t \ell(X_s) \, dB_s. \tag{2.6}$$

The second equality in (2.6) holds because we can (and do) take $\ell + \mathbf{P}^{\bullet}_{\text{op}}[1 - e^{-T_F}] = 1$ on E_Δ and $\ell = 0$ on $E_\Delta \setminus F$. Moreover,

$$\mathbf{P}^x_{\text{op}}[\Phi] = \frac{\mathbf{P}^x[\Phi]}{\mathbf{P}^x[1 - e^{-T_F}]}, \qquad \forall x \in E_\Delta \setminus F.$$

This choice of of $(\mathbf{P}^{\bullet}_{\text{op}}, B)$ having been made, we have

$$U^1_B 1_{E_\Delta}(x) = \mathbf{P}^x[\exp(-T_F)], \qquad \forall x \in E_\Delta. \tag{2.7}$$

A second key ingredient in our development is the *Lévy system* describing the totally inaccessible jumps of X. Recall that a stopping time T is totally inaccessible if $\mathbf{P}^x[T = S] = 0$ for all x and all predictable S. Let $(X^r_{t-})_{t>0}$ denote the left limit process of X, the limits being taken in some Ray-Knight compactification $\overline{E}$ of E_Δ; see [**Sh88**; §17–18]. The set

$$J := \{(\omega, t) : X^r_{t-}(\omega) \in E_\Delta, X^r_{t-}(\omega) \neq X_t(\omega)\} \tag{2.8}$$

is the union $\cup_n [\![T_n]\!]$ of a sequence of totally inaccessible stopping times. Indeed, a stopping time T is totally inaccessible if and only if $[\![T]\!] \subset J$ up to evanescence. Also, if we write X_{t-} for the left limit of X at time $t > 0$ (in the original topology of E) whenever it exists, then

$$J \subset \{(\omega, t) : X_{t-}(\omega) = X^r_{t-}(\omega)\} \tag{2.9}$$

up to evanescence; see [**Sh88**; (46.3)]. The Lévy system consists of a kernel N_Δ from $(E, \mathcal{E})$ to $(E_\Delta, \mathcal{E}_\Delta)$ such that $N_\Delta(x, \{x\}) = 0$ for all $x \in E$, and a CAF H, such that

$$\mathbf{P}^x \sum_{s \in J} Z_s \Psi(X_{s-}, X_s) = \mathbf{P}^x \int_0^\infty Z_s \int_{E_\Delta} \Psi(X_s, y)\, N_\Delta(X_s, dy)\, dH_s, \tag{2.10}$$

for all predictable $Z \geq 0$, $\Psi \in p(\mathcal{E} \otimes \mathcal{E}_\Delta)$, and $x \in E_\Delta$. We will often write $N_\Delta(x, \Psi)$ for $\int_{E_\Delta} \Psi(x, y)\, N_\Delta(x, dy)$; with this notation the right side of (2.10) collapses to $\mathbf{P}^x \int_0^\infty Z_s\, N_\Delta(X_s, \Psi)\, dH_s$. Because X cannot jump out of Δ, we can (and do) assume that $H_t = H_\zeta$ for all $t > \zeta$. Because H is a (finite) CAF, there is a strictly positive function $g \in \mathcal{E}^e$ such that $\sup_x \mathbf{P}^x \int_0^\infty e^{-t} g(X_t)\, dH_t < \infty$. Therefore, at the cost of replacing H_t by $\int_0^t g(X_s)\, dH_s$ and $N_\Delta(x, y)$ by $g(x)^{-1} N_\Delta(x, dy)$, we can arrange for H to have a bounded 1-potential.

3. Predictable Exit System.

We now introduce a supplementary hypothesis under which the optional exit system $(\mathbf{P}^\bullet_{\text{op}}, B)$ can be replaced by a *predictable exit system* $(\mathbf{P}^\bullet_{\text{pr}}, C)$ more suited to the problem of Lévy systems and time changes.

The set G^i, defined in (2.3), of irregular left endpoints of the intervals contiguous to M can be expressed as the disjoint union $\cup_n [\![T_n]\!]$ of graphs of stopping times. The situation we have in mind is when each of these stopping times is totally inaccessible. To state the following characterization of the hypothesis to be employed, define

$$D_t := \inf\{s > t : X_s \in F\} = t + T_F \circ \theta_t, \qquad t \geq 0.$$

The process D is increasing and right continuous, and $M \setminus G = \{(\omega, t) : D_t(\omega) = t\}$.

(3.1) Lemma. *The following conditions are equivalent:*

- *(i)* $G^i \subset J$;
- *(ii) The dual predictable projection of the AF* B^d *(defined in (2.5)) is continuous;*
- *(iii) The* 1*-potential* $\varphi_1 : x \mapsto \mathbf{P}^x[\exp(-T_F)]$ *is regular.*

(iv) The process $t \mapsto D_t$ is quasi-left continuous.

Proof. (i)$\Longrightarrow$(ii). Let $B^{d,p}$ denote the dual predictable projection of B^d. If T is a predictable time, then

$$\begin{aligned}\mathbf{P}^x[B^{d,p}_T - B^{d,p}_{T-}, 0 < T < \infty] &= \mathbf{P}^x\left[\int_{]0,\infty[} 1_{[\![T]\!]}(s)\, dB^{d,p}_s\right] \\ &= \mathbf{P}^x\left[\int_{]0,\infty[} 1_{[\![T]\!]}(s)\, dB^{d}_s\right] \\ &= \mathbf{P}^x[1-\varphi_1(X_T); T \in G^i] = 0,\end{aligned}$$

for all $x \in E_\Delta$.

(ii)$\Longrightarrow$(iii) The process $t \mapsto e^{-t}\varphi_1(X_t)$ is a positive right-continuous supermartingale. Indeed, from (2.7),

$$e^{-t}\varphi_1(X_t) = \mathbf{P}^x\left[\int_{]t,\infty[} e^{-s}\, dB_s \Big| \mathcal{F}_t\right] = \mathbf{P}^x\left[\int_{]t,\infty[} e^{-s}\, dB^p_s \Big| \mathcal{F}_t\right], \qquad \forall x \in E_\Delta, \tag{3.2}$$

where B is the AF component of the optional exit system for F and B^p is the dual predictable projection of B. The hypothesis (ii) implies that B^p is continuous, which in turn implies that φ_1 is regular, because of (3.2).

(iii)$\Longrightarrow$(iv) Let (T_n) be an increasing sequence of stopping times with limit T, and set $\Upsilon :=\uparrow \lim_n D_{T_n} \le D_T$. Then, for $x \in E_\Delta$,

$$\begin{aligned}\mathbf{P}^x[\exp(-\Upsilon)] = \lim_n \mathbf{P}^x[\exp(-D_{T_n})] &= \lim_n \mathbf{P}^x[e^{-T_n}\varphi_1(X_{T_n})] \\ = \mathbf{P}^x[e^{-T}\varphi_1(X_T)] &= \mathbf{P}^x[\exp(-D_T)]\end{aligned}$$

the third equality resulting from the assumed regularity of φ_1. It follows that $\Upsilon = D_T$ almost surely.

(iv)$\Longrightarrow$(i) Let T be a stopping time with $[\![T]\!] \subset G^i$. Then, on $\{T < \infty\}$, we have $0 < T = D_{T-} < D_T$. The quasi-left-continuity of D now implies that T is totally inaccessible. Thus, by [**Sh88**; (44.5)], $[\![T]\!] \subset J$. ∎

(3.3) Remark. Combining (3.1) with [**Sh88**; (46.2)] we see that if $G^i \subset J$ then

$$G \subset \{(\omega, t) : X_{t-}(\omega) = X^r_{t-}(\omega) \in E_\Delta\}$$

up to evanescence. This observation will be used several times in the sequel.

We assume in the remainder of the paper that the equivalent conditions stated in Lemma (3.1) are satisfied. We then have a predictable exit system, as detailed in the following theorem; *cf.* [**Ma75**; §8]. As preparation we first express the dual predictable projection of B^d in terms of the Lévy system.

(3.4) Lemma. *We have, for predictable $Z \geq 0$, $\Psi \in p\mathcal{F}^*$, and $x \in E_\Delta$,*

$$\mathbf{P}^x \sum_{t \in G^i} Z_t \, \Psi \circ \theta_t = \mathbf{P}^x \int_0^\infty 1_F(X_t) Z_t \int_{F^c} N_\Delta(X_t, dy) \mathbf{P}^y[\Psi] \, dH_t. \tag{3.5}$$

Proof. Define $I_t := \limsup_{s \uparrow\uparrow t} 1_F(X_s)$; this is a predictable process by the discussion on pp. 202–203 of [**Sh88**] or by [**DM78**; T-IV90(a)]. Moreover, $G^i = J \cap \{(\omega, t) : I_t(\omega) = 1, X_t(\omega) \in F^c\}$, up to evanescence. Therefore

$$\begin{aligned} \mathbf{P}^\bullet \sum_{t \in G^i} Z_t \, \Psi \circ \theta_t &= \mathbf{P}^\bullet \sum_{t \in G^i} Z_t \mathbf{P}^{X_t}[\Psi] \\ &= \mathbf{P}^\bullet \sum_{t \in J} I_t Z_t 1_{F^c}(X_t) \mathbf{P}^{X_t}[\Psi] \\ &= \mathbf{P}^\bullet \int_0^\infty I_t Z_t \int_{F^c} N_\Delta(X_t, dy) \mathbf{P}^y[\Psi] \, dH_t \\ &= \mathbf{P}^\bullet \int_0^\infty 1_F(X_t) Z_t \int_{F^c} N_\Delta(X_t, dy) \mathbf{P}^y[\Psi] \, dH_t \end{aligned} \tag{3.6}$$

We have used the fact that $\{(\omega, t) : t > 0, I_t(\omega) = 1, X_t(\omega) \notin F\} \subset G$, which implies that the sets $\{t : I_t = 1\}$ and $\{t : X_t \in F\}$ differ by at most a countable set, almost surely. This difference is not charged by H. ∎

Define $\psi(x) := \mathbf{P}^x[1 - \exp(-T_F)]$ and then take $\Psi = \psi(X_0)$ in (3.6) to see that the dual predictable projection of B^d is

$$\int_0^t 1_F(X_s) \, N_\Delta(X_s, \psi) \, dH_s, \qquad t \geq 0. \tag{3.7}$$

Accordingly we define a CAF C by

$$C_t := B_t^c + \int_0^t 1_F(X_s) \, N_\Delta(X_s, \psi) \, dH_s, \qquad t \geq 0, \tag{3.8}$$

noting that the 1-potential of C is

$$U_C^1 1_{E_\Delta}(x) = \mathbf{P}^x \int_0^\infty e^{-t} \, dC_t = \mathbf{P}^x \int_0^\infty e^{-t} \, dB_t = \mathbf{P}^x[\exp(-T_F)], \tag{3.9}$$

for all $x \in E_\Delta$.

By Motoo's theorem there are positive $\mathcal{E}^e$-measurable functions b and h such that

$$B_t^c = \int_0^t b(X_s) \, dC_s \quad \text{and} \quad \int_0^t 1_F(X_s) \, N_\Delta(X_s, \varphi) \, dH_s = \int_0^t h(X_s) \, dC_s. \tag{3.10}$$

We may suppose that $b+h=1$ on F and that $b=h=0$ on $E_\Delta\setminus F$. Finally, define

$$\mathbf{P}^x_{\text{pr}}[\Phi] := b(x)\cdot\mathbf{P}^x_{\text{op}}[\Phi] + h(x)1_F(x)\cdot\frac{\int_{F^c}\mathbf{P}^y[\Phi]\,N_\Delta(x,dy)}{N_\Delta(x,\varphi)}, \tag{3.11}$$

the ratio on the right being taken to be 0 when the denominator vanishes. Notice that

$$\int_0^t 1_F(X_s)\,ds = \int_0^t \gamma(X_s)\,dC_s, \qquad \forall t\ge 0,$$

where $\gamma := \ell\cdot b$.

(3.12) Theorem. *The pair $(\mathbf{P}^\bullet_{\text{pr}}, C)$ is a predictable exit system for F, in the sense that C is a CAF with fine support F, and*

$$\mathbf{P}^\bullet\sum_{t\in G} Z_t\Psi\circ\theta_t = \mathbf{P}^\bullet\int_0^\infty Z_t\mathbf{P}^{X_t}_{\text{pr}}[\Psi]\,dC_t, \tag{3.13}$$

for all predictable $Z\ge 0$ and $\Psi\in p\mathcal{F}^*$. Conversely, if there is a predictable exit system for F, then the conditions listed in (3.1) hold.

Proof. From (3.5) we see that

$$\mathbf{P}^\bullet\sum_{t\in G^i} Z_t\Psi\circ\theta_t = \mathbf{P}^\bullet\int_0^\infty Z_t\mathbf{P}^{X_t}_0[\Psi]1_F(X_t)N_\Delta(X_t,\varphi)\,dH_t, \tag{3.14}$$

where

$$\mathbf{P}^x_0[\Psi] := \frac{\int_{F^c}N_\Delta(x,dy)\mathbf{P}^y[\Psi]}{N_\Delta(x,\varphi)},$$

with the understanding that the ratio vanishes when the denominator is zero. Combining this with (2.4), (3.10), and (3.11), we obtain (3.13).

To see that C has fine support equal to F, we use the fact (3.9) that $\mathbf{P}^x[\exp(-T_F)] = U^1_C 1_{E_\Delta}(x)$ for all x. Let $R_C := \inf\{t : C_t > 0\}$. Clearly $R_C\ge T_F$ because C is carried by F. On the other hand

$$\begin{aligned}\mathbf{P}^x[\exp(-T_F)] = U^1_C 1_{E_\Delta}(x) &= \mathbf{P}^x\int_0^\infty e^{-t}\,dC_t\\ &= \mathbf{P}^x\int_{R_C}^\infty e^{-t}\,dC_t\\ &= \mathbf{P}^x\left[\exp(-R_C)U^1_C 1_{E_\Delta}(X_{R_C})\right]\\ &\le \mathbf{P}^x[\exp(-R_C)],\end{aligned} \tag{3.15}$$

because $U^1_C 1_{E_\Delta}\le 1_{E_\Delta}$. Together with the previously noted inequality $R_C\ge T_F$, (3.15) implies that $R_C = T_F$ almost surely.

Suppose, conversely, that $(\mathbf{P}^\bullet_{\rm pr}, C)$ is a predictable exit system for F; that is, (3.13) holds. One readily checks that $\varphi_1 := \mathbf{P}^\bullet[e^{-T_F}]$ is the 1-potential of the CAF $t \mapsto \int_0^t 1_F(X_s)\,ds + \int_0^t \mathbf{P}_{\rm pr}^{X(s)}[1-e^{-T_F}]\,dC_s$, which implies that φ_1 is regular. $\square$

4. Time Change.

Recall from section 2 that A is a CAF of X with fine support F. Thus F is finely perfect and the closed visiting set M has ω-sections that are perfect (or empty) almost surely. Let $\tau = (\tau(t))_{t\ge 0}$ denote the right-continuous inverse of A:

$$\tau(t) := \inf\{s : A_s > t\}, \qquad t \ge 0. \tag{4.1}$$

Then τ is strictly increasing (while finite), and as t varies the path $t \mapsto \tau(t)$ traces out $M \setminus G$.

As is well known the time-changed process $\widetilde X$ defined by

$$\widetilde X_t := X_{\tau(t)}, \qquad t \ge 0, \tag{4.2}$$

(with the convention $\widetilde X_\infty = \Delta$) is a right process with state space F, though $\widetilde X$ need not be a *Borel* right process.

We note in passing that if a nearly Borel set $L \subset F$ is X-polar (that is, $\mathbf{P}^x[X_t \in L$ for some $t>0] = 0$ for all $x \in E$) then L is also $\widetilde X$-polar. Conversely, if $L \subset F$ is $\widetilde X$-polar, then L is X-semipolar. In fact, if L is $\widetilde X$-polar, then the visiting set $\{(\omega,t) : X_t(\omega) \in L\}$ is contained in the graph of T_F, up to evanescence. Indeed, since $L \subset F$, it is clear that $\{(\omega,t) : X_t(\omega) \in L\} \subset [\![T_F, \infty[\![$. In view of the observation made at the end of the first paragraph of this section, the $\widetilde X$-polarity of L implies that $\{(\omega,t) : X_t(\omega) \in L\} \subset G$. In particular, $\{t > 0 : X_t(\omega) \in L\}$ is countable, a.s. Thus, if we fix an initial distribution μ, then

$$\begin{aligned}\mathbf{P}^\mu \sum_{s\in G} 1_L(X_s)[1-\exp(-T_F\circ\theta_s)] &= \mathbf{P}^\mu \sum_{s\in G^r} 1_L(X_s)[1-\exp(-T_F\circ\theta_s)]\\ &= \mathbf{P}^\mu \int_0^\infty 1_L(X_s)\mathbf{P}_{\rm op}^{X_s}[1-\exp(-T_F)]\,dB^c_s = 0,\end{aligned}$$

because B^c is continuous. To see that, in general, L need not be X-polar, consider the example of X equal to uniform motion to the right on $\mathbf{R}$ with $F = [0,\infty[$ and $L = \{0\}$.

5. Lévy System for $\widetilde X$.

In this section we give an explicit description of the Lévy system of $\widetilde X$ in terms of the Lévy system of X and the predictable exit system for F. The key observation is the following lemma, in which $\widetilde{J}$ is defined for $\widetilde{X}$ in just the same way that J was defined for X. We write Λ^+ (resp. Λ^-) for the set of points of right (resp. left) increase of A:

$$\Lambda^+ := \{(\omega, t) : t \geq 0, A_t(\omega) < A_{t+\epsilon}(\omega), \forall \epsilon > 0\}, \tag{5.1}$$

$$\Lambda^- := \{(\omega, t) : t > 0, A_{t-\epsilon}(\omega) < A_t(\omega), \forall \epsilon > 0\}. \tag{5.2}$$

The set Λ^+ is progressively measurable; in fact,

$$\Lambda^+ = M \setminus G. \tag{5.3}$$

Consequently, by the strong Markov property and Blumenthal's zero-one law, if T is a stopping time, then $[\![T]\!] \subset \Lambda^+$ if and only if $X_T \in F$, almost surely. Meanwhile, with $I_t = \limsup_{s \uparrow\uparrow t} 1_{\{X_S \in F\}}$ as before,

$$\Lambda^- = \{(\omega, t) : I_t(\omega) = 1\}, \tag{5.4}$$

so that Λ^- is predictable.

(5.5) Lemma. *$\{\tau(t) : t \in \widetilde{J}, \tau(t-) = \tau(t)\} = J \cap \Lambda^- \cap \Lambda^+$, up to evanescence.*

Proof. Let ρ be a metric on E_Δ compatible with the Ray topology induced there by X. When viewed as a process with values in the metric space (E_Δ, ρ), X is a right process; consequently, $\widetilde{X}$ is a right process when viewed as a process with state space (F_Δ, ρ), where $F_\Delta := F \cup \{\Delta\}$. The corresponding Ray-Knight compactification $\overline{F}$ of F_Δ (determined by $\widetilde{X}$) induces a topology on F_Δ; let $\widetilde{\rho}$ be a metric compatible with that topology. We shall write $\widetilde{X}^r_{t-}$ for the left limit (in $\overline{F}$) of $\widetilde{X}$ at time $t > 0$. Thus,

$$\widetilde{J} = \{(\omega, t) : \widetilde{X}^r_{t-}(\omega) \in F_\Delta, \widetilde{X}^r_{t-}(\omega) \neq \widetilde{X}_t(\omega)\}.$$

We write $J^\#$ for $\{\tau(t) : t \in \widetilde{J}, \tau(t-) = \tau(t)\}$. It is clear that $J^\# \subset \Lambda^- \cap \Lambda^+$. Let $s := \tau(t) < \infty$ be a typical element of $J^\#$, so that $A_s = t$. In view of (2.9) applied to $\widetilde{X}$,

$$\rho\text{-}\lim_{u \uparrow t} \widetilde{X}_u = \widetilde{\rho}\text{-}\lim_{u \uparrow t} \widetilde{X}_u = \widetilde{X}^r_{t-}.$$

But, because X has left limits in the ρ topology and $\tau(t-) = \tau(t)$,

$$\begin{aligned} \rho\text{-}\lim_{u \uparrow t} \widetilde{X}_u &= \rho\text{-}\lim_{u \uparrow t} X_{\tau(u)} \\ &= X^r_{\tau(t-)-} = X^r_{\tau(t)-} = X^r_{s-}. \end{aligned}$$

Hence $X^r_{s-} = \widetilde{X}^r_{t-} \neq \widetilde{X}_t = X_s$, from which we deduce that $s \in J$. This proves that $J^\# \subset J \cap \Lambda^- \cap \Lambda^+$.

For the reverse containment we begin by observing that $J \cap \Lambda^+ \cap \Lambda^-$ is an $(\mathcal{F}_t)$-progressively measurable subset of the $(\mathcal{F}_t)$-optional set $J \cap \Lambda^-$ ($\Lambda^+ = M \setminus G$ is $(\mathcal{F}_t)$-progressive) and has countable sections. Thus, by [**DM78**; TIV.88], there is a sequence (S_n) of $(\mathcal{F}_t)$-stopping times such that $J \cap \Lambda^+ \cap \Lambda^- = \cup_n [\![S_n]\!]$ up to evanescence. The containment at issue will therefore be established once we show that if S is an $(\mathcal{F}_t)$-stopping time with $[\![S]\!] \subset J \cap \Lambda^+ \cap \Lambda^-$ then $[\![S]\!] \subset J^\#$. Fix such a stopping time S and define $T := A_S$. Notice that $\tau(T-) = \tau(T) = S$ almost surely on $\{S < \infty\} = \{S < \zeta\}$, because $[\![S]\!] \subset \Lambda^+ \cap \Lambda^-$. We are going to show that T is a totally inaccessible stopping time of the filtration $\widetilde{\mathcal{F}}_t := \mathcal{F}_{\tau(t)}$, $t \geq 0$. It is clear that T is an $(\widetilde{\mathcal{F}}_t)$-stopping time because $\{T \leq t\} = \{A_S \leq t\} = \{S \leq \tau(t)\} \in \mathcal{F}_{\tau(t)}$. Suppose there exist $x \in F$ and an $(\widetilde{\mathcal{F}}_t)$-predictable time R such that $\mathbf{P}^x[T = R < \infty] > 0$. We claim that $\tau(R-)$ is an $(\mathcal{F}_t)$-predictable time. Indeed, let (R_n) announce R; see [**DM78**; IV.77], and note that $\widetilde{\mathcal{F}}_t$ is contained in the $\mathbf{P}^x$-completion of $\mathcal{F}^\circ$. Then

$$\{\tau(R-) \leq t\} = \{R \leq A_t\} = \cap_n \{R_n < A_t\},$$

and

$$\{R_n < A_t\} = \cup_{q \in \mathbf{Q}} \{R_n < q < A_t\} = \cup_{q \in \mathbf{Q}} \{R_n < q, \tau(q) < t\} \in \mathcal{F}_t,$$

since $\{R_n < q\} \in \mathcal{F}_{\tau(q)}$. Consequently, $\tau(R-)$ is an $(\mathcal{F}_t)$-stopping time. This same argument applies to each R_n. But $t \mapsto \tau(t-)$ is strictly increasing on $]0, A_\zeta]$ and identically infinite on $]A_\zeta, \infty[$. It follows that the sequence $(\tau(R_n-) \wedge n)$ of $(\mathcal{F}_t)$ stopping times announces $\tau(R-)$. But on the event $\{T = R < \infty\}$, we have $S = \tau(T-) = \tau(R-)$, contradicting the total inaccessibility of S. ∎

Define

$$\widetilde{C}_t := C_{\tau(t)}, \qquad \widetilde{H}^F_t := \int_0^{\tau(t)} 1_F(X_s)\, dH_s. \tag{5.6}$$

Because the fine support of C is F while that of $t \mapsto \int_0^t 1_F(X_s)\, dH_s$ is contained in F, both $\widetilde{C}$ and $\widetilde{H}^F$ are CAFs of $\widetilde{X}$. Now define

$$\widetilde{H}_t := \widetilde{C}_t + \widetilde{H}^F_t, \qquad t \geq 0. \tag{5.7}$$

Another application of Motoo's theorem yields the existence of $\mathcal{E}^e$-measurable densities $\widetilde{c}$ and $\widetilde{h}$ (vanishing off F) such that

$$\widetilde{C}_t = \int_0^t \widetilde{c}(\widetilde{X}_s)\, d\widetilde{H}_s, \quad \text{and} \quad \widetilde{H}^F_t = \int_0^t \widetilde{h}(\widetilde{X}_s)\, d\widetilde{H}_s, \qquad \forall t \geq 0, \tag{5.8}$$

almost surely. Finally, define a kernel $\widetilde{N}_\Delta$ on $(F_\Delta, \mathcal{E}_\Delta \cap F_\Delta)$ by

$$\widetilde{N}_\Delta(x, dy) := 1_{F \times F_\Delta}(x, y) \left[\widetilde{c}(x)\, \mathbf{P}^x_{\mathrm{pr}}[X_{T_F} \in dy] + \widetilde{h}(x)\, N_\Delta(x, dy) \right]. \tag{5.9}$$

(5.10) Theorem. *The pair $(\widetilde{N}_\Delta, \widetilde{H})$ is a Lévy system for the totally inaccessible jumps of $\widetilde{X}$.*

Proof. Fix an $\widetilde{X}$-predictable process $\widetilde{Z} \geq 0$, and a positive Borel function Φ on the product space $F \times F_\Delta$. Using [**DM78**; (IV.67.1)] it is not hard to check that $s \mapsto \widetilde{Z}_{A(s)}$ is X-predictable. Then, using Lemma (5.5) for the first equality,

$$\begin{aligned}
\mathbf{P}^\bullet \sum_{t \in \widetilde{J}, \tau(t-) = \tau(t)} \widetilde{Z}_t \Phi(\widetilde{X}_{t-}, \widetilde{X}_t) &= \mathbf{P}^\bullet \sum_{s \in J} 1_{\Lambda^-}(s) \widetilde{Z}_{A(s)} \Phi(X_{s-}, X_s) 1_{\Lambda^+}(s) \\
&= \mathbf{P}^\bullet \sum_{s \in J} 1_{\Lambda^-}(s) \widetilde{Z}_{A(s)} \Phi(X_{s-}, X_s) 1_F(X_s) \\
&= \mathbf{P}^\bullet \int_0^\infty 1_{\Lambda^-}(s) \widetilde{Z}_{A(s)} \int_F N_\Delta(X_s, dy) \Phi(X_s, y)\, dH_s \\
&= \mathbf{P}^\bullet \int_0^\infty 1_F(X_s) \widetilde{Z}_{A(s)} \int_F N_\Delta(X_s, dy) \Phi(X_s, y)\, dH_s.
\end{aligned}$$

The second equality above follows from the discussion just after (5.3) because J is the disjoint union of graphs of stopping times, while the final equality holds because Λ^- differs from $\{X \in F\}$ by a countable set not charged by H. Consequently,

$$\mathbf{P}^\bullet \sum_{t \in \widetilde{J}, \tau(t-) = \tau(t)} \widetilde{Z}_t\, \Phi(\widetilde{X}_{t-}, \widetilde{X}_t) = \mathbf{P}^\bullet \int_0^\infty \widetilde{Z}_t \int_F N_\Delta(\widetilde{X}_t, dy)\, \Phi(\widetilde{X}_t, y)\, d\widetilde{H}^F_t. \tag{5.11}$$

Let us now turn to the jumps of $\widetilde{X}$ associated with the exits of X from F. Observe that if $t \in \widetilde{J}$ and $\tau(t-) < \tau(t)$, then $[\![\, \tau(t-) \,]\!] \subset G$, $\widetilde{X}_{t-}$ exists, and

$$\widetilde{X}_{t-} = \lim_{u \uparrow \tau(t-), u \in M \setminus G} X_u.$$

But by [**Sh88**; (46.2)], the set

$$\Gamma := \{(\omega, t) : X_{t-}(\omega) \text{ does not exist in } E_\Delta\}$$

is the union of graphs of a sequence of predictable stopping times, and so $\Gamma \cap G$ is evanescent in view of Lemma (3.1). It follows that

$$\widetilde{X}_{t-} = X_{\tau(t-)-}, \qquad \forall t \in \widetilde{J} \text{ such that } \tau(t-) < \tau(t),$$

almost surely. Therefore

$$
\begin{aligned}
\mathbf{P}^\bullet \sum_{t\in\widetilde{J},\tau(t-)<\tau(t)} & \widetilde{Z}_t\,\Phi(\widetilde{X}_{t-},\widetilde{X}_t) \\
&= \mathbf{P}^\bullet \sum_{s\in G} \widetilde{Z}_{A(s)}\,\Phi(X_{s-},X_{D_s}) \\
&= \mathbf{P}^\bullet \int_0^\infty \widetilde{Z}_{A(s)} \int_\Omega \Phi(X_s, X_{T_F}(\omega))\,\mathbf{P}^{X_s}_{\mathrm{pr}}(d\omega)\,dC_s \\
&= \mathbf{P}^\bullet \int_0^\infty \widetilde{Z}_t \int_\Omega \Phi(\widetilde{X}_t, X_{T_F}(\omega))\,\mathbf{P}^{\widetilde{X}_t}_{\mathrm{pr}}(d\omega)\,d\widetilde{C}_t.
\end{aligned}
\tag{5.12}
$$

Taken together, (5.11) and (5.12) imply that $(\widetilde{N}_\Delta,\widetilde{H})$ is a Lévy system for the totally inaccessible jumps of $\widetilde{X}$.

6. Jump Measures and Feller Measures.

We now fix an *excessive measure* m to serve as background measure. Thus m is a σ-finite measure on $(E,\mathcal{E})$ such that $mP_t(L)\le m(L)$ for all $L\in\mathcal{E}$ and $t>0$. Because (P_t) is a right semigroup, we then have $mP_t\uparrow m$ (setwise) as $t\downarrow 0$; see [**DM87**; XII 36-37]. Here and in the remainder of the paper the (absorbing) state Δ is viewed as a cemetery state; the stopping time $\zeta:=\inf\{t: X_t=\Delta\}$ is the *lifetime* of X. Accordingly, functions (resp. measures) defined on E (resp. $\mathcal{E}$) are extended to E_Δ (resp. $\mathcal{E}_\Delta$) by letting the value at Δ (resp. $\{\Delta\}$) be 0.

Let $R=(R_t)_{t\ge 0}$ be a raw (*i.e.*, not necessarily adapted) additive functional (RAF) of X. The Revuz measure of R, relative to m, is defined by the monotone limit

$$
\nu_R^m(f) :=\uparrow \lim_{t\downarrow 0} t^{-1}\mathbf{P}^m \int_0^t f(X_s)\,dR_s, \qquad f\in p\mathcal{E}^*. \tag{6.1}
$$

If R is a CAF then the measure ν_R^m is σ-finite, and two CAFs with the same Revuz measure are $\mathbf{P}^m$-indistiguishable. See [**G90**], [**FG96**], and [**FG03**] for more details. The "local" formula (6.1) defining ν_R^m has a "global" counterpart expressed in terms of the Kuznetsov process $((Y_t)_{t\in\mathbf{R}},Q_m)$ associated with X and m. The sample space for Y is W, the space of all paths $w:\mathbf{R}\to E_\Delta:=E\cup\{\Delta\}$ that are right continuous and E-valued on an open interval $]\alpha(w),\beta(w)[$ and take the value Δ outside of this interval. The dead path $[\Delta]$, constantly equal to Δ, corresponds to the interval being empty; by convention $\alpha([\Delta])=+\infty$, $\beta([\Delta])=-\infty$. The σ-algebra $\mathcal{G}^\circ$ on W is generated by the coordinate maps $Y_t(w)=w(t)$, $t\in\mathbf{R}$, and $\mathcal{G}_t^\circ:=\sigma(Y_s: s\le t)$. The Kuznetsov measure $\mathbf{Q}_m$ is the unique σ-finite measure on $\mathcal{G}^\circ$ not charging $\{[\Delta]\}$ such that, for $-\infty<t_1<t_2<\cdots<t_n<+\infty$,

$$
\begin{aligned}
\mathbf{Q}_m(Y_{t_1}\in dx_1, & Y_{t_2}\in dx_2,\ldots,Y_{t_n}\in dx_n) \\
&= m(dx_1)\,P_{t_2-t_1}(x_1,dx_2)\cdots P_{t_n-t_{n-1}}(x_{n-1},dx_n).
\end{aligned}
\tag{6.2}
$$

Because the only times appearing on the right side of (6.2) are the differences $t_k - t_{k-1}$, the measure $\mathbf{Q}_m$ is invariant with respect to the shift operators σ_t, $t \in \mathbf{R}$, defined by

$$\sigma_t w(s) = [\sigma_t w](s) := w(t+s), \qquad s \in \mathbf{R};$$

that is

$$\mathbf{Q}_m[\Phi \circ \sigma_t] = \mathbf{Q}_m[\Phi], \qquad \forall \Phi \in p\mathcal{G}^o, t \in \mathbf{R}.$$

It will be convenient to take $X = (X_t, \mathbf{P}^x)$ to be the realization of (P_t) described on p. 53 of [**G90**]. The sample space for X is

$$\Omega := \{\alpha = 0, Y_{\alpha+} \text{ exists in } E\} \cup \{[\Delta]\},$$

X_t is the restriction of Y_t to Ω for $t > 0$, and X_0 is the restriction of Y_{0+}. Moreover, $\mathcal{F}^\circ := \sigma(X_t : t \geq 0)$ is the trace of $\mathcal{G}^\circ$ on Ω.

To discuss the strong Markov property of Y, as well as the moderate markov property of Y when time is reversed, we recall the modified process Y^* of [**G90**; (6.12)]. Let d be a totally bounded metric on E compatible with the topology of E, and let $\mathcal{D}$ be a countable uniformly dense subset of the d-uniformly continuous bounded real-valued functions on E. Given a strictly positive $h \in b\mathcal{E}$ with $m(h) < \infty$ define $W(h) \subset W$ by the conditions:

(6.3) (i) $\alpha \in \mathbf{R}$;

(ii) $Y_{\alpha+} := \lim\limits_{t \downarrow \alpha} Y_t$ exists in E;

(iii) $U^q g(Y_{\alpha+1/n}) \to U^q g(Y_{\alpha+})$ as $n \to \infty$,
for all $g \in \mathcal{D}$ and all rationals $q > 0$;

(iv) $Uh(Y_{\alpha+1/n}) \to Uh(Y_{\alpha+})$ as $n \to \infty$.

Evidently $\sigma_t^{-1}(W(h)) = W(h)$ for all $t \in \mathbf{R}$, and $W(h) \in \mathcal{G}^\circ_{\alpha+}$ since E is a Lusin space. We now define

$$Y_t^*(w) = \begin{cases} Y_{\alpha+}(w), & \text{if } t = \alpha(w) \text{ and } w \in W(h), \\ Y_t(w), & \text{otherwise.} \end{cases} \tag{6.4}$$

(If h' is another function with the properties of h then $\mathbf{Q}_m(W(h) \triangle W(h')) = 0$.)

The process Y^* features in a maximal form of the the strong Markov property, recorded in (6.5) below; for a proof see [**G90**; (6.15)]. (This process will also be used in section 7 to define the moderate Markov dual of X with respect to m.) A clean statement of this result requires the "truncated shift" operators θ_t, $t \in \mathbf{R}$ defined by

$$\theta_t w(s) = [\theta_t w](s) := \begin{cases} w(t+s), & s > 0; \\ \Delta, & s \leq 0. \end{cases}$$

The filtration $(\mathcal{G}_t^m)_{t \in \mathbf{R}}$ is obtained by augmenting $(\mathcal{G}_t^\circ)_{t \in \mathcal{R}}$ with the $\mathbf{Q}_m$ null sets in the usual way.

(6.5) Proposition. *Let T be a $(\mathcal{G}_t^m)$-stopping time. Then $\mathbf{Q}_m$ restricted to $\mathcal{G}_T^m \cap \{Y_T^* \in E\}$ is a σ-finite measure and*

$$\mathbf{Q}_m(F \circ \theta_T \mid \mathcal{G}_T^m) = P^{Y_T^*}(F), \quad \mathbf{Q}_m\text{-a.e.} \quad on \quad \{Y_T^* \in E\}$$

for all $F \in p\mathcal{F}^\circ$.

Now given an RAF R there is a uniquely determined (up to $\mathbf{Q}_m$ evanescence) homogeneous random measure (HRM) κ_R such that

$$\kappa_R]s,t] = R_{t-s} \circ \theta_s, \qquad \text{on } \{\alpha < s < \beta\}, \mathbf{Q}_m\text{-a.s.},$$

for all real $s < t$. The global counterpart to (6.1) that was alluded to earlier is this:

$$\text{(6.6)} \qquad \mathbf{Q}_m \int_R f(Y_t, t)\, \kappa_R(dt) = \int_E \int_{\mathbf{R}} f(x,t)\, dt\, \nu_R^m(dx), \qquad \forall f \in p(\mathcal{E} \otimes \mathcal{B}).$$

See [**G90**; (8.21), (8.26)].

As an example, let us consider additive functionals related to the Lévy system (N_Δ, H) discussed at the end of section 2. As is customary, we now break N_Δ into two pieces

$$\text{(6.7)} \qquad N(x, dy) := 1_E(y) N_\Delta(x, dy), \qquad n(x) := N_\Delta(x, \{\Delta\}),$$

and define the "killing rate" CAF K by

$$\text{(6.8)} \qquad K_t := \int_0^t n(X_s)\, dH_s, \qquad t \geq 0.$$

Taking $Z = 1_{]0,t]}$ in (2.10) we find that

$$\text{(6.9)} \qquad \mathbf{P}^x \sum_{s \in J, s \leq t} \Psi(X_{s-}, X_s) 1_{\{s < \zeta\}} = \mathbf{P}^x \int_0^t N(X_s, \Psi)\, dH_s, \qquad x \in E, t \geq 0,$$

and

$$\text{(6.10)} \qquad \mathbf{P}^x[f(X_{\zeta-}); \zeta \leq t, \zeta \in J] = \mathbf{P}^x \int_0^t f(X_s)\, dK_s,$$

for $\Psi \in p(\mathcal{E} \otimes \mathcal{E})$ and $f \in p\mathcal{E}$. It follows from this discussion and (6.1) that

$$\text{(6.11)} \qquad \mathcal{J}(\Psi) :=\uparrow \lim_{t \downarrow 0} t^{-1} \mathbf{P}^m \sum_{s \in J, s \leq t} \Psi(X_{s-}, X_s) 1_{\{s < \zeta\}} = \nu_H^m(N\Psi),$$

and

$$\mathcal{K}(f) :=\uparrow \lim_{t\downarrow 0} t^{-1}\mathbf{P}^m[f(X_{\zeta-}); \zeta \le t, \zeta \in J] = \nu_K^m(f) = \nu_H^m(nf). \tag{6.12}$$

That is, the “jump measure” $\mathcal{J}$ is given on $E \times E$ by the formula

$$\mathcal{J}(dx, dy) = \nu_H^m(dx) N(x, dy), \tag{6.13}$$

while the “killing measure” $\mathcal{K}$ is given on E by

$$\mathcal{K}(dx) = n(x)\,\nu_H^m(dx). \tag{6.14}$$

Now let J^* denote the set of totally inaccessible jumps of Y (defined as J was for X). Paralleling (2.9) we have (employing the obvious notation regarding left limits)

$$J^* \subset \{(w,t) \in W \times \mathbf{R} : \alpha(w) < t < \beta(w), Y_{t-}(w) = Y^r_{t-}(w)\} \tag{6.15}$$

up to $\mathbf{Q}_m$-evanescence. Combining (6.6) with the version of (2.10) valid for Y we now obtain

$$\begin{aligned} \mathbf{Q}_m \sum_{s\in J^*} f(t, Y_{t-}, Y_t) &= \mathbf{Q}_m \int_{\mathbf{R}}\int_E f(t, Y_t, y)\, N(Y_t, dy)\, \kappa_H(dt) \\ &= \int_{\mathbf{R}} dt \int_{E\times E} f(t,x,y)\, \mathcal{J}(dx, dy), \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} \mathbf{Q}_m[f(\beta, Y_{\beta-}); \beta \in J^*] &= \mathbf{Q}_m \int_{\mathbf{R}} f(t, Y_t) n(Y_t)\, \kappa_H(dt) \\ &= \int_{\mathbf{R}} dt \int_E f(t,x)\, \mathcal{K}(dx). \end{aligned} \tag{6.17}$$

We record the analogous results for the optional and predictable exit systems for F. The “optional” version comes from [**FM86**] (see also [**G90**; (11.6)]; the “predictable” version is proved in a similar manner. Let F be a finely perfect nearly Borel set. For $w \in W$ let $M^*(w)$ be the closure in $\mathbf{R}$ of $\{t \in \mathbf{R} : Y_t(w) \in F\}$ and let $G^*(w)$ be the set of left endpoints (in $]\alpha(w), \infty[$) of the contiguous intervals of $M^*(w)$. It is readily verified that if $\alpha < s < t$ then $t \in M^*$ if and only if $t - s \in M \circ \theta_s$, and likewise for G^*. Moreover, under the conditions listed in (3.1), the left limit Y_{s-} exists in E and is equal to the Ray left limit Y^r_{s-}, for all $s \in G^*$, $\mathbf{Q}_m$-a.s.

(6.18) Proposition. *Let* $(\mathbf{P}^\bullet_{\mathrm{op}}, B)$ *and* $(\mathbf{P}^\bullet_{\mathrm{pr}}, C)$ *be optional and predictable exit systems for* F*, respectively. Let* $\nu_B = \nu^m_B$ *and* $\nu_C = \nu^m_C$ *be the corresponding Revuz measures, with respect to* m*. Then* ν_B *and* ν_C *are* σ*-finite, and*

$$\mathbf{Q}_m \sum_{s\in G^*} f(s, Y_s, \theta_s) = \int_{\mathbf{R}} dt \int_E \nu_B(dx)\, \mathbf{P}^x_{\mathrm{op}}[f(t,x,\cdot)], \tag{6.19}$$

$$\mathbf{Q}_m \sum_{s\in G^*} f(s, Y_{s-}, \theta_s) = \int_{\mathbf{R}} dt \int_E \nu_C(dx)\, \mathbf{P}^x_{\mathrm{pr}}[f(t,x,\cdot)], \tag{6.20}$$

provided $f \in p(\mathcal{B}\otimes\mathcal{E}^*_\Delta\otimes\mathcal{F}^*)$.

(6.21) Remark. It is worth noting that (6.19) and subsequent expressions involving the optional exit system are valid even when the conditions listed in (3.1) do not hold.

Following [**CFY06b**] (see also [**FHY04**] and [**CFY06a**]) we define the *Feller measure*

$$\Lambda(\Gamma) :=\uparrow \lim_{t\downarrow 0} t^{-1}\mathbf{P}^m \sum_{s\in G} 1_{]0,t]}(s) 1_\Gamma(X_{s-}, X_{D_s}) 1_{\{D_s<\infty\}}, \qquad \Gamma\in\mathcal{E}\otimes\mathcal{E}, \tag{6.22}$$

and the *supplementary Feller measure*

$$\delta(L) :=\uparrow \lim_{t\downarrow 0} t^{-1}\mathbf{P}^m \sum_{s\in G} 1_{]0,t]}(s) 1_L(X_{s-}) 1_{\{D_s=\infty\}}, \qquad L\in\mathcal{E}. \tag{6.23}$$

Since $X_{D_s} = X_{T_F}\circ\theta_s$, the exit system formula (3.11) implies that

$$\mathbf{P}^m \sum_{s\in G} 1_{]0,t]}(s) 1_\Gamma(X_{s-}, X_{D_s}) 1_{\{D_s<\infty\}} = \mathbf{P}^m \int_0^t \phi(X_s)\, dC_s, \tag{6.24}$$

where

$$\phi(x) := \mathbf{P}^x_{\mathrm{pr}}[1_\Gamma(x, X_{T_F}); T_F<\infty], \qquad x\in E.$$

Formula (6.24) yields the existence of the monotone limit in (6.22) and even identifies the limit as $\nu_C(\phi)$. Hence,

$$\Lambda(\Gamma) = \nu_C(\phi) = \int_F \nu_C(dx)\, \mathbf{P}^x_{\mathrm{pr}}[1_\Gamma(x, X_{T_F}); T_F<\infty]. \tag{6.25}$$

Similar considerations lead to the identification of the limit in (6.23) as

$$\delta(L) = \int_L \nu_C(dx)\, \mathbf{P}^x_{\mathrm{pr}}[T_F=\infty]. \tag{6.26}$$

The next two formulas follow immediately from Proposition (6.20) and formulas (6.25) and (6.26).

(6.27) Proposition. *For $\Phi \in p(\mathcal{B}\otimes\mathcal{E}\otimes\mathcal{E})$ and $f \in p(\mathcal{B}\otimes\mathcal{E})$,*

$$\mathbf{Q}_m \sum_{t\in G^*} \Phi(t, Y_{t-}, Y_{D_t}) 1_{\{D_t<\infty\}} = \int_{\mathbf{R}} dt \int_{F\times F} \Phi(t,x,y)\Lambda(dx,dy), \tag{6.28}$$

and

$$\mathbf{Q}_m \sum_{t\in G^*} f(t, Y_{t-}) 1_{\{D_t=\infty\}} = \int_{\mathbf{R}} \int_F f(t,x)\, dt\, \mathbf{P}^x_{\mathrm{pr}}[T_F=\infty]\nu_C(dx). \tag{6.29}$$

Recall from section 5 the CAF A, its inverse τ, and the time-changed process $\widetilde{X} = X_\tau$. We are now going to exhibit formulas for the jump and killing measures of $\widetilde{X}$, in terms of the corresponding measures for X and the exit system for F. In the course of the proof we shall use the following result taken from section 6 of [**FG88**].

(6.30) Lemma. *Let $\widetilde{m}$ denote ν_A^m, and let Z be a CAF of X with fine support contained in F (That is, $Z_{T_F} = 0$, a.s.) Then $\widetilde{m}$ is $\widetilde{X}$-excessive, $\widetilde{Z}_t := Z_{\tau(t)}$, $t \ge 0$, defines a CAF of $\widetilde{X}$, and the Revuz measure $\widetilde{\nu}_{\widetilde{Z}}^{\widetilde{m}}$ of $\widetilde{Z}$, relative to $\widetilde{m}$, is equal to ν_Z^m.*

The following result extends [**CFY06b**; Thm. 5.6] to the context of this paper.

(6.31) Theorem. *The jump measure $\widetilde{\mathcal{J}}$ and the killing measure $\widetilde{\mathcal{K}}$ for the time-changed process $\widetilde{X}$, with respect to the $\widetilde{X}$-excessive measure $\widetilde{m} := \nu_A^m$, are given respectively by the formulas*

$$\widetilde{\mathcal{J}} = 1_{F\times F}\,\mathcal{J} + \Lambda, \tag{6.32}$$

$$\widetilde{\mathcal{K}} = 1_F\,\mathcal{K} + \delta. \tag{6.33}$$

Proof. We prove only (6.32), as the proof of (6.33) is quite similar. Let Γ be a Borel measurable subset of $F\times F$, and let us begin with

$$\mathbf{P}^x \sum_{s\in\widetilde{J}, s\le t} 1_\Gamma(\widetilde{X}_{s-}, \widetilde{X}_s) = \mathbf{P}^x \int_0^t \widetilde{N}(\widetilde{X}_s, 1_\Gamma)\, d\widetilde{H}_s, \qquad x\in F. \tag{6.34}$$

In view of (5.6)–(5.9), the right side of (6.34) is equal to

$$\mathbf{P}^x \int_0^t N(\widetilde{X}_s, 1_\Gamma)\, d\widetilde{H}_s^F + \mathbf{P}^x \int_0^t \phi(X_s)\, d\widetilde{C}_s,$$

where $\phi(x) := \mathbf{P}^x_{\rm pr}[1_\Gamma(x, X_{T_F}); T_F < \infty]$. By (5.6) and Lemma (6.30) the Revuz measure of $\widetilde{H}^F$ (relative to $\widetilde{m} := \nu^m_A$) is $1_F \nu^m_H$, while that of $\widetilde{C}$ is ν^m_C. Therefore

$$
\begin{aligned}
\widetilde{\mathcal{J}}(\Gamma) &= \widetilde{\nu}^{\widetilde{m}}_{\widetilde{H}}(\widetilde{N} 1_\Gamma) \\
&= \int_{F\times F} \nu^m_H(dx)\, N(x, dy) + \int_{F\times F} \nu_C(dx)\, \mathbf{P}^x_{\rm pr}[X_{T_F} \in dy, T_F < \infty].
\end{aligned}
\tag{6.35}
$$

□

7. Entrance Law.

Because of the time-symmetry of the Markov property, the process $(Y_t, \mathbf{Q}_m)$ is a Markov process with respect to the reverse filtration $\widehat{\mathcal{G}}_t := \sigma\{Y_s : s \ge t\}$, $t \in \mathbf{R}$. Unlike the situation in "forward" time, this process need not be a strong Markov process, but it is a *moderate* Markov process. To make this precise we define

$$\widehat{\Omega} := \{\beta = 0\} \subset W; \tag{7.1}$$

$$\widehat{X}_t(\hat\omega) := Y^*_{-t}(\hat\omega), \qquad t > 0, \hat\omega \in \widehat{\Omega} \tag{7.2}$$

$$\widehat{\mathcal{F}}_t := \sigma\{\widehat{X}_s : 0 < s \le t\}, t > 0, \quad \widehat{\mathcal{F}} := \sigma\{\widehat{X}_s : s > 0\} \tag{7.3}$$

$$\check{\theta}_t w(s) := \begin{cases} w(t-s), & s > 0; \\ \Delta, & s \le 0. \end{cases} \tag{7.4}$$

Then there is a Borel measurable family $\{\widehat{\mathbf{P}}^x, x \in E\}$ of probability measures on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ under which $(\widehat{X}_t)_{t>0}$ has the moderate Markov property:

$$\widehat{\mathbf{P}}^x[f(\widehat{X}_{T+s}) | \widehat{\mathcal{F}}_{T-}] = \widehat{\mathbf{P}}^{\widehat{X}_T}[f(\widehat{X}_s)], \qquad s > 0, f \in b\mathcal{E}, \tag{7.5}$$

whenever T is an $(\widehat{\mathcal{F}}_t)$-predictable stopping time. (As a matter of convention, $\mathbf{P}^x[\widehat{X}_0 = x] = 1$ and $\widehat{\mathcal{F}}_{0-} = \{\emptyset, \widehat{\Omega}\}$.) The measures $\widehat{\mathbf{P}}^x$ are uniquely determined modulo an m-polar set. (A set $L \in \mathcal{E}^e$ is m-polar provided $\mathbf{P}^m[T_L < \infty] = 0$.) The link between Y and $\widehat{X}$ is this: If $T : W \to [-\infty, \infty]$ is $(\widehat{\mathcal{G}}_t)$-predictable, then for $\Phi \in p\mathcal{F}$,

$$\mathbf{Q}_m[\Phi \circ \check{\theta}_T | \widehat{\mathcal{G}}_{T-}] = \widehat{\mathbf{P}}^{Y^*_T}[\Phi], \qquad \text{on } \{Y^*_T \in E\}, \tag{7.6}$$

the σ-finiteness of $\mathbf{Q}_m$ on $\widehat{\mathcal{G}}_{T-} \cap \{Y^*_T \in E\}$ being part of the assertion. For more details see [**G99**; §2], [**Fi87**; §4], and [**Ma93**].

It follows easily from (7.6) (with T a fixed time) that the transition semigroup $(\widehat{P}_t)$ of $\widehat{X}$, defined by $\widehat{P}_t f = \widehat{\mathbf{P}}^\bullet[f(\widehat{X}_t)]$ is in duality with (P_t) with respect to m:

$$(P_t f, g) = (f, \widehat{P}_t g), \qquad f, g \in \mathcal{E}^*, t > 0, \tag{7.7}$$

in which $(f,g) := \int_E fg\, dm$ provided the integral exists. Likewise, defining the associated resolvent

$$\widehat{U}^\lambda f = \int_0^\infty e^{-\lambda t} \widehat{P}_t f\, dt = \widehat{\mathbf{P}}^\bullet \int_0^\infty e^{-\lambda t} f(\widehat{X}_t)\, dt, \tag{7.8}$$

we have

$$(U^\lambda f, g) = (f, \widehat{U}^\lambda g), \qquad f, g \in \mathcal{E}^*, \lambda > 0. \tag{7.9}$$

We usually omit the hat $\widehat{\ }$ in those places where it is obviously required. For example, we write $\widehat{\mathbf{P}}^\bullet[f(X_t)]$ in place of $\widehat{\mathbf{P}}^\bullet[f(\widehat{X}_t)]$.

Before proceeding, we collect some facts about the moderate Markov dual process. Recall that a set $L \in \mathcal{E}^e$ is m-semipolar provided the visiting set $\{t > 0 : X_t \in L\}$ is $\mathbf{P}^m$-a.s. at most countable. Also, property $P(x)$ depending on $x \in E$ is said to hold m-quasi-everywhere (m-q.e.) provided $\{x \in E : P(x) \text{ fails}\}$ is m-polar. Define

$$\widehat{T}_F := \inf\{t \in]0, \widehat{\zeta}[: \widehat{X}_t \subset F\}.$$

(7.10) Lemma. *Let F be a finely perfect nearly Borel subset of E.*

- *(i)* $\{x \in F : \widehat{\mathbf{P}}^x[T_F = 0] < 1\}$ *is m-semipolar.*
- *(ii)* $\{x \in E \setminus F : \widehat{\mathbf{P}}^x[T_F = 0] > 0\}$ *is m-semipolar.*
- *(iii)* $t \mapsto \widehat{X}_t$ *has right limits in $\overline{E}$ (with respect to the Ray topology) on* $[0, \infty[$, $\widehat{\mathbf{P}}^x$*-a.s. for m-q.e. $x \in E$.*

Proof. (i) Let μ be a finite measure on E not charging m-semipolar sets. Then there is a diffuse optional copredictable HRM κ with Revuz measure μ; see [**Fi87**; (5.22)] or [**FG06**; (3.10)]. Let ϕ be a strictly positive Borel function on $\mathbf{R}$ with $\int_{\mathbf{R}} \phi(t)\, dt = 1$. Since κ is copredictable,

$$\begin{aligned} \mathbf{Q}_m \int_{\mathbf{R}} \phi(t) 1_F(Y_t^*) 1_{\{\widehat{T}_F \circ \check{\theta}_t = 0\}}\, \kappa(dt) &= \mathbf{Q}_m \int_{\mathbf{R}} \phi(t) 1_F(Y_t^*) \widehat{P}^{Y^*(t)}[\widehat{T}_F = 0]\, \kappa(dt) \\ &= \int_F \widehat{\mathbf{P}}^x[T_F = 0]\, \mu(dx). \end{aligned} \tag{7.11}$$

Let Z denote the closure of $\{t \in]\alpha, \beta[: Y_t \in F\}$. Then

$$Z \cap \{t : \widehat{T}_F \circ \check{\theta}_t > 0\} = Z \cap \{t : \exists \epsilon > 0,]t - \epsilon, t[\cap Z = \emptyset\}.$$

Hence, $Z \cap \{t : \widehat{T}_F \circ \check{\theta}_t > 0\}$ is contained in the set of right endpoints of the contiguous intervals of Z. But there are only countably many such intervals, and κ is diffuse, so

$$\mathbf{Q}_m \int_{\mathbf{R}} \phi(t) 1_F(Y_t^*) 1_{\{\widehat{T}_F \circ \check{\theta}_t = 0\}} \, \kappa(dt) = \mu(F). \tag{7.12}$$

It follows that $\{x \in F : \widehat{\mathbf{P}}^x[T_F = 0] < 1\}$ has μ-measure equal to 0. Since μ was an arbitrary finite measure not charging m-semipolars, a result of Dellacherie [**De88**; p. 70] tells us that $\{x \in F : \widehat{\mathbf{P}}^x[T_F = 0] < 1\}$ is m-semipolar.

(ii) Using the notation established in the proof of point (i),

$$\mathbf{Q}_m \int_{\mathbf{R}} \phi(t) 1_{\{\widehat{T}_F \circ \check{\theta}_t = 0\}} 1_{E \setminus F}(Y_t^*) \, \kappa(dt) = \int_{E \setminus F} \widehat{\mathbf{P}}^x[T_F = 0] \, \mu(dx). \tag{7.13}$$

If $\widehat{T}_F \circ \check{\theta}_t = 0$ then for every sufficiently small $\eta > 0$ the interval $]t - \eta, t[$ contains times at which Y is in F; if also $Y_t \in E \setminus F$ then t is an element of G^* because $E \setminus F$ is finely open. Since κ is diffuse, the above displayed integrals must vanish. Point (ii) now follows as did (i).

(iii) By considering $f(\hat{X}_t)$ as f runs through a countable dense subset of $C(\overline{E})$ one sees that the set of $(\hat{\omega}, t)$ such that $s \mapsto \widehat{X}_s(\widehat{\omega})$ fails to have a right limit in $\overline{E}$ at t is $(\widehat{\mathcal{F}}_t^P)_{t \ge 0}$-progressively measurable. Here P is an arbitrary probability measure on $(\widehat{\Omega}, \widehat{\mathcal{F}}^\circ)$, and $(\widehat{\mathcal{F}}_t^P)$ is the usual right-continuous completion of $(\widehat{\mathcal{F}}_t^o)$. See, for example, [**DM78**; IV-90]. It follows that the projection Π of the above-described set onto $\widehat{\Omega}$ is an element of $\widehat{\mathcal{F}}^* := \cap_P \widehat{\mathcal{F}}^P$. Note that Π is the set of $\widehat{\omega}$ for which $s \mapsto \widehat{X}_s(\widehat{\omega})$ fails to have a right limit in $\overline{E}$ at some $t > 0$. Hence, $f(x) := \widehat{\mathbf{P}}^x[\Pi]$ is $\mathcal{E}^*$-measurable, and then f is coexcessive. Since $\widehat{X}_s \circ \check{\theta}_t = Y_{t-s}$ for $s > 0$, the set $\check{\theta}_t^{-1}\Pi$ is contained in the set of $w \in W$ such that $r \mapsto Y_r(w)$ fails to have a left limit in $\overline{E}$ at some $r \in]-\infty, t[$, and so $\mathbf{Q}_m[\check{\theta}_t^{-1}\Pi] = 0$ for all $t \in \mathbf{R}$. Now

$$m(f) = \mathbf{Q}_m \left[\widehat{\mathbf{P}}^{Y(0)}[\Pi] \right] = \mathbf{Q}_m[\check{\theta}_0^{-1}\Pi, \alpha < 0 < \beta] = 0.$$

Hence $f = 0$, m-a.e., and therefore $\widehat{\mathbf{P}}^x[\Pi] = f(x) = 0$ for m-q.e. $x \in E$; see [**G99**; (2.11)]. □

Define, for $\lambda \ge 0$ and $f \in p\mathcal{E}^*$,

$$\widehat{P}_F^\lambda f(x) := \widehat{\mathbf{P}}^x[e^{-\lambda T_F} f(X_{T_F})] \tag{7.14}$$

$$\widehat{P}_{F+}^\lambda f(x) := \widehat{\mathbf{P}}^x[e^{-\lambda T_F} f(X_{T_F+})], \tag{7.15}$$

with the understanding that $\exp(-0\cdot\infty)=0$, so that $\widehat{P}_F f := \widehat{P}^0_F f = \widehat{\mathbf{P}}^x[f(X_{T_F}) : T_F < \infty]$. Here $\widehat{X}_{T_F+}$ denotes the right limit (in $\overline{E}$ with its Ray topology) of $t \mapsto \widehat{X}_t$ at $\widehat{T}_F$. In (7.15), f is extended to all of $\overline{E}$ by declaring $f(x)=0$ for $x \in \overline{E}\setminus E$. In the light of (7.10)(iii), $\widehat{X}_{T_F+}$ exists $\widehat{\mathbf{P}}^x$-a.s. on $\{\widehat{T}_F < \infty\}$ for m-q.e. $x \in E$. Thus both $\widehat{P}^\lambda_{F+} f$ and $\widehat{P}^\lambda_F f$ are uniquely determined m-q.e. and are $\mathcal{E}^*$-measurable.

Recall the optional and predictable exit systems $(\mathbf{P}^\bullet_{\text{op}}, B)$ and $(\mathbf{P}^\bullet_{\text{pr}}, C)$ for F. Since the measure m will remain fixed in the sequel, we shall write ν_B for ν^m_B and ν_C for ν^m_C. The *balayage* of m on F is the excessive measure $R_F m$ defined by

$$R_F m(f) := \mathbf{Q}_m[f(Y_t); T_F < t], \qquad f \in p\mathcal{E}. \tag{7.16}$$

Here $T_F := \inf\{t \in]\alpha,\beta[: Y_t \in F\}$ extends the previously defined hitting time of F to all of W. Upon noting that $f(Y_t)1_{\{T_F<t\}} = \left[f(Y_0)1_{\{T_F<0\}}\right]\circ\sigma_t$, it becomes clear that the right side of (7.16) does not depend on t. Moreover, because $\{T_F < 0\} = \check{\theta}_0^{-1}\{\widehat{T}_F < \widehat{\zeta}\}$, we see from (7.16) that

$$R_F m(f) = \mathbf{Q}_m\left[f(Y_0)\widehat{\mathbf{P}}^{Y_0}[\widehat{T}_F < \widehat{\zeta}]\right] = (\widehat{P}_F 1, f),$$

because $\widehat{\mathbf{P}}^x[T_F < \zeta] = \widehat{P}_F 1(x)$ for all $x \in E$. It is important to note at this stage that

$$\widehat{P}_{F+}1(x) = \widehat{P}_F 1(x), \qquad \text{for } m\text{-a.e.} x \in E. \tag{7.17}$$

(In fact, for m-q.e. x, but the m-a.e. assertion is sufficient for our purposes.) To see this fix a strictly positive $f \in b\mathcal{E}$ with $m(f) < \infty$, and define $G_0 := \sup\{t \le 0 : Y_t \in F\}$. Observe that $0 \le \widehat{P}_F 1(x) - \widehat{P}_{F+}1(x) = \widehat{\mathbf{P}}^x[T_F < \zeta, X_{T_F+} \notin E]$. Using (7.6) with $T=0$ we have

$$\begin{aligned}\int_E f(x)\left[\widehat{P}_F 1(x) - \widehat{P}_{F+}1(x)\right] m(dx) &= \mathbf{Q}_m[f(Y_0); \alpha < G_0, Y^r_{G_0-} \notin E]\\ &= \mathbf{Q}_m[f(Y_0); \alpha < G_0, Y_{G_0-} = \Delta]\\ &= 0,\end{aligned}$$

the second equality following from Remark (3.3) reinterpreted for Y, and the third from the fact that Δ is isolated in E_Δ.

The following decomposition of $R_F m$ (for general Borel F) appears in section 6 of [**FM86**]:

$$(\widehat{P}_F 1, f) = R_F m(f) = \nu_B(\ell f + V^0_{\text{op}} f), \qquad f \in p\mathcal{E},$$

where $V^0_{\text{op}} f(x) := \mathbf{P}^x_{\text{op}} \int_0^{T_F} f(X_t)\,dt$. Our principal goal in the remainder of this section is to generalize this decomposition and to obtain its predictable analog.

(7.18) Notation. $(f,g)_0 := \int_{E\setminus F} fg\,dm$.

(7.19) Theorem. *If $f,g \in p\mathcal{E}^*$, then*

(i) $(\widehat{P}^\lambda_F f, g)_0 = \nu_B(f\cdot V^\lambda_{\rm op} g)$,

(ii) $(\widehat{P}^\lambda_{F+} f, g)_0 = \nu_C(f\cdot V^\lambda_{\rm pr} g)$.

Here $V^\lambda_{\rm op} f = \mathbf{P}^\bullet_{\rm op}\int_0^{T_F} e^{-\lambda t} f(X_t)\,dt$ *and* $V^\lambda_{\rm pr} f$ *is defined analogously with* $\mathbf{P}^\bullet_{\rm pr}$ *replacing* $\mathbf{P}^\bullet_{\rm op}$.

Proof. We shall prove only (ii), as the proof of (i) is similar but easier since no Ray limit is involved. From (7.10)(ii), $\widehat{\mathbf{P}}^x[T_F > 0] = 1$, m-a.e. on $E\setminus F$. Thus

$$\begin{aligned}(\widehat{P}^\lambda_{F+} f, g)_0 &= \mathbf{Q}_m[\widehat{P}^\lambda_{F+} f(Y_0) g(Y_0); Y_0 \in E\setminus F]\\ &= \mathbf{Q}_m\left[e^{-\lambda \widehat{T}_F\circ\check{\theta}_0} f(\widehat{X}^r(\widehat{T}_{F+}))\circ\check{\theta}_0\, g(Y_0); \widehat{T}_F\circ\check{\theta}_0 > 0, Y_0\in E\setminus F\right],\end{aligned}$$

because $\widehat{T}_F\circ\check{\theta}_0 > 0$ on the event $\{Y_0 \in E\setminus F\}$. If $s = -\widehat{T}_F\circ\check{\theta}_0 > \alpha$, then $s\in G^*$ (defined below (6.17)), and $]s, s+T_F\circ\theta_s[$ is the unique interval contiguous to M^* that contains 0. If $s\in G^*$, then $Y^r_{s-} = Y_{s-}$ by Remark (3.3). Therefore, using (6.20) for the second equality below,

$$\begin{aligned}(\widehat{P}^\lambda_{F+} f, g)_0 &= \mathbf{Q}_m \sum_{s\in G^*, s<0} e^{\lambda s} f(Y_{s-}) g(X_{-s})\circ\theta_s\, 1_{\{s+T_F\circ\theta_s>0\}}\\ &= \int_0^\infty e^{-\lambda t}\,dt \int_F \nu_C(dx) f(x)\mathbf{P}^x_{\rm pr}[g(X_t); t<T_F]\\ &= \nu_C(f\cdot V^\lambda_{\rm pr} g).\end{aligned}$$

□

Let $(Q_t)_{t\ge0}$ and $(V^\lambda)_{\lambda\ge0}$ denote the semigroup and resolvent for (X, T_F), the process X killed at time T_F:

$$Q_t f(x) := \mathbf{P}^x[f(X_t); t<T_F], \quad V^\lambda f(x) := \mathbf{P}^x\int_0^{T_F} e^{-\lambda t} f(X_t)\,dt = \int_0^\infty e^{-\lambda t} Q_t f(x)\,dt.$$

Let $(\widehat{X}, \widehat{T}_F)$ denote $\widehat{X}$ killed at $\widehat{T}_F$, with corresponding semigroup $(\widehat{Q}_t)_{t\ge0}$ and resolvent $(\widehat{V}^\lambda)_{\lambda\ge0}$. As is customary, $V := V^0$ and $\widehat{V} := \widehat{V}^0$. It is known that (X, T_F) and $(\widehat{X}, \widehat{T}_F)$ are dual processes, in the sense that $(V^\lambda f, g)_0 = (f, \widehat{V}^\lambda g)_0$ for all $f,g\in p\mathcal{E}^*$ and $\lambda\ge 0$. See [**FG06**; (A.7)]. We write $Q^{\rm op}_t f := \mathbf{P}^\bullet_{\rm op}[f(X_t); t<T_F]$, with an analogous definition for $Q^{\rm pr}_t$.

(7.20) Corollary. *Fix $f\in bp\mathcal{E}^*$. Then the formulas*

$$\eta^f(g) := (\widehat{P}_{F+} f, g)_0 \quad\text{and}\quad \xi^f(g) := (\widehat{P}_F f, g)_0, \qquad g\in p\mathcal{E}, \tag{7.21}$$

define (σ-finite) purely excessive measures η^f and ξ^f for (X, T_F). Moreover, $\eta_t^f := (f\nu_C)Q_t^{\mathrm{pr}}$ and $\xi_t^f := (f\nu_B)Q_t^{\mathrm{op}}$, $t > 0$, define entrance laws for (X, T_F) such that

$$\eta^f = \int_0^\infty \eta_t^f \, dt \quad \text{and} \quad \xi^f = \int_0^\infty \xi_t^f \, dt.$$

If, in addition, $\nu_C(f) < \infty$ (resp. $\nu_B(f) < \infty$), then η_t^f (resp. ξ_t^f) is a finite measure for each $t > 0$.

Proof. Clearly η^f and ξ^f are σ-finite measures on $E \setminus F$. If $\lambda > 0$ then $\eta^f(V^\lambda g) = (\widehat{P}_{F+}f, V^\lambda g)_0 = (\widehat{V}^\lambda \widehat{P}_{F+}f, g)_0$, while

$$\lambda \widehat{V}^\lambda \widehat{P}_{F+}f = \widehat{\mathbf{P}}^\bullet[(1 - e^{-\lambda T_F}) f(X_{T_F+}) : T_F < \infty] \downarrow 0,$$

as $\lambda \downarrow 0$, since f is bounded. If $0 < g \le 1$ on $E \setminus F$ and $m_0(g) < \infty$, then $\lambda \eta^f(V^\lambda g) \downarrow 0$ as $\lambda \downarrow 0$, and so η^f is a purely excessive measure for (X, T_F). It follows from (7.19)(ii) that $\eta^f = \int_0^\infty \eta_t^f \, dt$. Using the fact that $(X_t)_{t>0}$ under $\mathbf{P}_{\mathrm{pr}}^x$ is Markovian with transition semigroup (P_t), one easily checks that $\eta_{t+s}^f = \eta_t^f Q_s$ for $t, s > 0$. Recall that $\mathbf{P}_{\mathrm{pr}}^\bullet[1 - \exp(-T_F)] \le 1$; see (3.11) and the sentence following (2.6). Now $(1 - e^{-t}) \le (1 - e^{-T_F})$ on $\{t < T_F\}$, and ν_C is σ-finite. This implies that η_t^f is a countable sum of finite measures for each $t > 0$. Fix $g \in p\mathcal{E}$ with $0 < g \le 1$ on $E \setminus F$ and $\eta^f(g) < \infty$. Then $Vg > 0$ on $E \setminus F$, and we may use the Fubini theorem to conclude that

$$\eta_t^f(Vg) = \int_t^\infty \eta_s^f(g) \, ds \le \eta^f(g) < \infty.$$

Therefore η_t^f is in fact σ-finite for each $t > 0$. Consequently, $(\eta_t^f)_{t>0}$ is an entrance law for (X, T_F). If, in addition, $\nu_C(f) < \infty$, then $\eta^f(1) \le \nu_C(f) < \infty$. The treatment of $(\xi_t^f)_{t>0}$ is similar. ∎

(7.22) Corollary. *For $f, g \in \mathcal{E}^*$,*

(i) $(\widehat{P}_F^\lambda f, g) = \nu_B(\ell f g) + \nu_B(f V_{\mathrm{op}}^\lambda g) = \nu_{B^c}(\ell f g) + \nu_B(f V_{\mathrm{op}}^\lambda g)$;

(ii) $(\widehat{P}_{F+}^\lambda f, g) = \nu_C(\gamma g \widehat{P}_{0+} f) + \nu_C(f V_{\mathrm{pr}}^\lambda g)$.

Here ℓ comes from (2.6), B^c is the continuous part of B, γ is defined just below (3.11), and $\widehat{P}_{0+}f := \widehat{\mathbf{P}}^\bullet[f(X_{0+})]$.

Proof. Since $\sigma_t \mathbf{Q}_m = \mathbf{Q}_m$ for all $t \in \mathbf{R}$,

$$\begin{aligned}\int_F \widehat{P}_F^\lambda f(x) g(x) \, m(dx) &= \mathbf{Q}_m[\widehat{P}_F^\lambda f(Y_0) g(Y_0) 1_F(Y_0)] \\ &= \int_0^1 \mathbf{Q}_m[\widehat{P}_F^\lambda f(Y_t) g(Y_t) 1_F(Y_t)] \, dt.\end{aligned}$$

Also, (2.6) implies that $1_F(Y_t)\,dt = \ell(Y_t)\,\kappa_B(dt)$, where κ_B is the HRM of Y that extends B; notice that κ_B has Revuz measure ν_B. See, for example, the discussion on pages 89–91 of [**G90**]. Therefore, by [**G90**; (8.21)],

$$\begin{aligned}\int_F \widehat{P}_F^\lambda f(x) g(x)\,m(dx) &= \mathbf{Q}_m \int_0^1 \widehat{P}_F^\lambda f(Y_t) g(Y_t)\ell(Y_t)\,\kappa_B(dt)\\ &= \nu_B(\ell g \widehat{P}_F^\lambda f).\end{aligned}$$

But $\ell = 0$ on $E\setminus F$ and $\ell\nu_B = \ell\nu_{B^c}$. In view of (7.10)(i), $\widehat{\mathbf{P}}^x[T_F = 0] = 1$, ν_{B^c}-a.e. because ν_{B^c} doesn't charge m-semipolars. Hence $\nu_B(\ell g\widehat{P}_F^\lambda f) = \nu_{B^c}(\ell g f)$, since $\widehat{\mathbf{P}}^x[\widehat{X}_0 = x] = 1$ by convention. Combining this with (7.19)(i) yields (7.22)(i). A similar argument shows that

$$\int_F \widehat{P}_{F+}^\lambda f(x) g(x)\,m(dx) = \nu_C(\gamma g \widehat{P}_{F+}^\lambda f) = \nu_C(\gamma g \widehat{P}_{0+} f),$$

establishing (7.22)(ii). ∎

(7.23) Proposition. *If $f, g \in p\mathcal{E}^*$, then*

$$(\widehat{P}_{F+}^\lambda f, P_F g) = \nu_C(\gamma g \widehat{P}_{0+} f) + \nu_C(f V_{\rm pr}^\lambda P_F g) = (\widehat{P}_{F+} f, P_F^\lambda g). \tag{7.24}$$

Proof. The first equality is an immediate consequence of (7.22)(ii) since $P_F g = g$ on F. For the second equality, arguing as in the proof of (7.22), we have

$$\int_F \widehat{P}_F f \cdot P_F^\lambda g\,dm = \nu_C(\gamma g \widehat{P}_{0+} f).$$

Also, as in the proof of (7.19),

$$\begin{aligned}(\widehat{P}_{F+} f, P_F^\lambda g)_0 &= \mathbf{Q}_m \sum_{s\in G^*, s<0} f(Y_{s-}) e^{-\lambda(s+T_F\circ\theta_s)} g(X_{T_F})\circ\theta_s 1_{\{s+T_F\circ\theta_s>0\}}\\ &= \int_F \nu_C(dx) f(x) \mathbf{P}_{\rm pr}^x \int_{-\infty}^0 e^{-\lambda(s+T_F)} g(X_{T_F}) 1_{\{T_F>-s\}}\,ds\\ &= \int_F \nu_C(dx) f(x) \mathbf{P}_{\rm pr}^x \int_0^{T_F} e^{-\lambda(T_F-s)} g(X_{T_F})\,ds\\ &= \int_F \nu_C(dx) f(x) \mathbf{P}_{\rm pr}^x \int_0^{T_F} e^{-\lambda u} g(X_{T_F})\,du\\ &= \int_F \nu_C(dx) f(x) \mathbf{P}_{\rm pr}^x \int_0^{T_F} e^{-\lambda u} g(X_{T_F})\circ\theta_u, du\\ &= \nu_C(f V_{\rm pr}^\lambda P_F g).\end{aligned}$$

Combining these observations yields the second equality in (7.24). ∎

In the same way one has

$$(\widehat{P}_F^\lambda f, P_F g) = \nu_{B^c}(\ell f g) + \nu_B(f V_{\text{op}}^\lambda g) = (\widehat{P}_F f, P_F^\lambda g). \tag{7.25}$$

Let us suppose in this paragraph that $R_F m = m$ (otherwise, replace m with $R_F m$.) Let us make the special choice $A = C$ in the preceding discussion. Then by (7.17) and (7.22)(ii) with $\lambda = 0$ and $f = 1$,

$$m(g) = \nu_C^m(\gamma g + V_{\text{pr}} g), \qquad g \in p\mathcal{E}, \tag{7.26}$$

where $V_{\text{pr}} = V_{\text{pr}}^0$. Notice that the right side of (7.26) depends on m only through the Revuz measure ν_C^m, which is excessive for the time-changed process $\widetilde{X}$. Following up on earlier work ([**Ha56, Ka83, Ka84, FG06, Si80...**] we use this formula to construct an excessive measure for X, given an excessive measure for $\widetilde{X}$.

(7.27) Proposition. *Suppose that* $\mathbf{P}^x[T_F < \infty] > 0$ *for all* $x \in E$. *Let* ν *be an excessive measure for* $\widetilde{X}$. *Then*

$$\eta(g) := \nu(\gamma g + V_{\text{pr}} g), \qquad g \in p\mathcal{E} \tag{7.28}$$

defines an excessive measure for X *such that (i)* $R_F \eta = \eta$ *and (ii)* $\nu_C^\eta = \nu$. *The measure* η *is uniquely determined by these two conditions.*

Proof. According to [**FG88**; (5.12), (5.13)], under the hypothesis of the proposition there is a uniquely determined X-excessive measure $m = m^\nu$ such that $R_F m = m$ and $\nu_C^m = \nu$. By (7.22)(ii) we have

$$m(g) = R_F m(g) = (\widehat{P}_F 1, g) = \nu_C^m(\gamma g + V_{\text{pr}} g) = \nu(\gamma g + V_{\text{pr}} g).$$

The right side of (7.28) therefore defines an excessive measure for X with the stated properties. ∎

Define a measure Θ on $F \times F \times]0, \infty[$ by

$$\Theta(dx, dy, dt) := \nu_C(dx) \mathbf{P}_{\text{pr}}^x [X_{T_F} \in dy, T_F \in dt]. \tag{7.29}$$

Because ν_C is σ-finite and $0 < \mathbf{P}_{\text{pr}}^\bullet[1 - \exp(-T_F)] \le 1$, it is easy to check that Θ is σ-finite. Notice that the Feller measure Λ is related to Θ by

$$\Lambda(dx, dy) = \Theta(dx, dy,]0, \infty[).$$

Intuitively, $\Theta(dx, dy, dt)$ is the rate at which excursions from F of duration t originate from x and terminate at y.

(7.30) Proposition. *Suppose $f, g \in p(\mathcal{E}^* \cap F)$ with $\nu_C(f) < \infty$ and g bounded. Then $\Theta(f, g, dt)$ is a σ-finite measure on $]0, \infty[$, and for $\lambda > 0$,*

$$\lambda(\widehat{P}^\lambda_{F+} f, P_F g) = \lambda\nu_C(\gamma g \widehat{P}_{0+} f) + \int_0^\infty (1 - e^{-\lambda t})\, \Theta(f, g, dt). \tag{7.31}$$

Proof. It is immediate that $\int_0^\infty (1 - e^{-t})\, \Theta(f, g, dt)$ is dominated by $\|g\|_\infty \nu_C(f)$. Consequently, $\Theta(f, g, \cdot)$ is σ-finite. Then

$$\begin{aligned}\int_0^\infty (1 - e^{-\lambda t})\, \Theta(f, g, dt) &= \int_F f(x) \mathbf{P}^x_{\rm pr}[g(X_{T_F})(1 - e^{-\lambda T_F})]\, \nu_C(dx) \\ &= \lambda \int_F f(x) \nu_C(dx) \mathbf{P}^x_{\rm pr} \int_0^{T_F} P_F g(X_t) e^{-\lambda t}\, dt \\ &= \lambda \nu_C(f V^\lambda_{\rm pr} P_F g),\end{aligned}$$

and combining this with (7.22)(ii) we obtain (7.31). ∎

(7.32) Remarks. Analogous results hold for the optional exit system. For example, employing the obvious notation,

$$\lambda(\widehat{P}^\lambda_F f, P_F g) = \lambda \nu_{B^c}(\gamma f g) + \int_0^\infty \Theta_{\rm op}(f, g, dt).$$

As pointed out earlier, $\eta^f_t(g) = \nu_C(f Q^{\rm pr}_t g)$ and $\xi^f_t(g) = \nu_B(f Q^{\rm op}_t g))$ are entrance laws for (X, T_F) provided $f \in bp\mathcal{E}^*$. If $f \equiv 1$ then $\widehat{P}^\lambda_{F+} 1 = \widehat{P}^\lambda_F 1 = \widehat{\mathbf{P}}^\bullet[e^{-\lambda T_F}]$, m-a.e. by the obvious variant of (7.17). Let us write $\widehat{\varphi}$ for this last function when $\lambda = 0$. Theorem (7.19) implies that $\nu_C V^\lambda_{\rm pr} = \nu_B V^\lambda_{\rm op}$ as measures on $E \setminus F$ for all $\lambda \geq 0$. In particular, $\eta^1_t = \xi^1_t$ since either entrance law integrates to the measure $\widehat{\varphi} m_0$, which is purely excessive for (X, T_F). (Here $m_0 := m|_{E\setminus F}$.)

We conclude by recording extensions to the present context of some formulas obtained in [**FG06**] in the context of excursions from a point. First recall the definition of the energy functional L^0 of the killed process $X^0 := (X, T_F)$; see [**G90**; §3]. If ξ is an X^0-excessive measure and f is an X^0-excessive function, then

$$L^0(\xi, f) := \sup\{\mu(f) : \mu V \leq \xi\}, \tag{7.33}$$

in which μ ranges over the σ-finite measures on F. If ξ is purely excessive for X^0, then [**G90**; (3.6)]

$$L^0(\xi, f) = \lim_{\lambda \to \infty} \lambda \langle \xi - \lambda \xi V^\lambda, f \rangle = \lim_{t \downarrow 0} t^{-1} \langle \xi - \xi Q_t, f \rangle, \tag{7.34}$$

where $\langle \mu, f \rangle := \int f\, d\mu$. (Both of the limits in (7.34) are monotone increasing.) Define

$$\psi := 1 - P_F 1 = \mathbf{P}^\bullet[T_F = \infty]. \tag{7.35}$$

It is easily checked that ψ is X^0-excessive. We fix $f \in bp\mathcal{E}$. Then $\eta^f := \widehat{P}_{F+} f \cdot m_0 = \int_0^\infty \eta^f_t\, dt$ is purely excessive for X^0.

(7.36) Theorem. *(i) If g is X^0-excessive then* $L^0(\eta^f, g) = \lim_{t\downarrow 0} \eta_t^f(g)$.

(ii) $L^0(\eta^f, \psi) = \int_F f(x)\mathbf{P}_{\mathrm{pr}}^x[T_F = \infty, \zeta > 0]\,\nu_C(dx)$.

(iii) (Recall the definition (6.6) of the supplementary Feller measure δ*.)*

$$\delta(f) = \int_F f(x)\mathbf{P}_{\mathrm{pr}}^x[T_F = \infty]\nu_C(dx) = L^0(\eta^f, \psi) + \int_F f(x)\mathbf{P}_{\mathrm{pr}}^x[\zeta = 0]\,\nu_C(dx). \tag{7.37}$$

Proof. Abbreviate $\eta = \eta^f$ and $\eta_t = \eta_t^f$ during this proof. Suppose first that $g \in p\mathcal{E}^*$ with $\eta(g) < \infty$. Then

$$\langle \eta - \eta Q_t, g\rangle = \eta(g) - \eta Q_t g = \int_0^t \eta_s(g)\,ds.$$

The extreme terms in this display are positive measures in g, so for general $g \in p\mathcal{E}^*$ we deduce that

$$\langle \eta - \eta Q_t, g\rangle = \int_0^t \eta_s(g)\,ds. \tag{7.38}$$

If g is X^0-excessive then $\eta_{t+s}(g) = \eta_t(Q_s g) \uparrow \eta_t(g)$ as $s \downarrow 0$. Thus $t \mapsto \eta_t(g)$ is right continuous and decreasing on $]0,\infty[$. In particular, $\uparrow \lim_{t\downarrow 0} \eta_t(g)$ exists, though it may equal $+\infty$. Therefore, by (7.38),

$$\begin{aligned} L^0(\eta, g) &= \lim_{t\downarrow 0} t^{-1}\int_0^t \eta_s(g)\,ds = \lim_{t\downarrow 0}\int_0^1 \eta_{tu}(g)\,du \\ &= \lim_{t\downarrow 0} \eta_t(g), \end{aligned}$$

by monotone convergence, establishing (i).

Next, $\eta_t(\psi) = \nu_C(fQ_t^{\mathrm{pr}}\psi)$ and $Q_t^{\mathrm{pr}}\psi = \mathbf{P}_{\mathrm{pr}}^{\bullet}[T_F = \infty; t < T_F \wedge \zeta] \uparrow \mathbf{P}_{\mathrm{pr}}^{\bullet}[T_F = \infty, \zeta > 0]$ as $t \downarrow 0$. Hence $L^0(\eta, \psi) = \mathbf{P}_{\mathrm{pr}}^{f\nu_C}[T_F = \infty, \zeta > 0]$. But $\mathbf{P}_{\mathrm{pr}}^{f\nu_C}[T_F = \infty, \zeta = 0] = \mathbf{P}_{\mathrm{pr}}^{f\nu_C}[\zeta = 0]$, proving both (ii) and (iii). ∎

(7.39) Remarks. Intuitively, $\delta(f) = \mathbf{P}_{\mathrm{pr}}^{f\nu_C}[T_F = \infty]$ represents the rate (weighted by f) at which a final excursion of infinite length occurs, terminating M. Theorem (7.36) indicates that $\mathbf{P}_{\mathrm{pr}}^{f\nu_C}[\zeta = 0]$ is the weighted rate at which the process X is killed while in F; $L^0(\eta, \psi)$ is the corresponding rate of occurrence of an excursion in which the process wanders away from F, never to return. Exactly the same argument establishes the analogous facts in the optional case.

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