On the Approximation of Markov Processes
by Compound Poisson Processes

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My intention in this note is to present a simple approach to the approximation of a continuous time Markov process by “compound Poisson” (or “pure jump”) processes. Such approximations have been discussed recently in quite general contexts in [5] and [4]. Our setting is broader than that found in these references, and our proof more direct. A related, but much deeper, approximation of diffusion processes in Euclidean space is the subject of [6].

Let \( X = (X_t, P^x) \) be a simple Markov process with right-continuous left-limited (rcll) sample paths, state space \( E \), transition semigroup \( (P_t)_{t \geq 0} \), and resolvent \( (U^\alpha)_{\alpha > 0} \). Thus, the action of \( P_t \) on a bounded Borel function \( f : E \to \mathbb{R} \) is given by

\[
P_tf(x) = P^x[f(X_t)], \quad t \geq 0, x \in E,
\]

where \( P^x \) is the law of \( X \) under the initial condition \( X_0 = x \); and

\[
U^\alpha f(x) := P^x \left[ \int_0^\infty e^{-\alpha t} f(X_t) \, dt \right] = \int_0^\infty e^{-\alpha t} P_tf(x) \, dt.
\]

So as not to obscure the main idea, we assume for simplicity that \( E \) is homeomorphic to a Borel subset of a compact metric space, and that \( P_tf \) is Borel measurable whenever the bounded function \( f \) is so. From this it follows that \( U^\alpha \) also preserves Borel measurability.

For each positive integer \( n \), the kernel \( K_n(x, dy) := nU^n(x, dy) \) is a probability kernel on \( E \); evidently \( K_n(x, \cdot) \) is the \( P^x \)-distribution of \( X_T \), where \( T \) is independent of \( X \) and exponentially distributed with rate \( n \). Let \( (Y^n(k) : k = 0, 1, 2, \ldots) \) be a Markov chain on \( E \) with one-step transition kernel \( K_n \). Let \( \Pi^n = (\Pi^n(t))_{t \geq 0} \) be a Poisson process of rate \( n \), independent of \( Y^n \), and define a continuous-time pure jump Markov process by

\[
Z^n_t := Y^n(\Pi^n(t)), \quad t \geq 0.
\]

The construction of such processes is discussed in detail in section of [2]. (In particular, \( Z^n \) is a strong Markov process.) The following result is proved in [5] for reversible Markov processes and in [4] for Hunt processes.
**Theorem.** If \( Y^n(0) = x \), then the sequence \( \{Z^n\}_{n \geq 1} \) of rcll processes converges weakly to \( X \) (under \( P^x \)) as \( n \to \infty \).

**Proof.** Fix \( x \in E \); in the following discussion, the law \( P^x \) will govern \( X \). Let \( \Lambda^n = (\Lambda^n(t))_{t \geq 0} \) be a Poisson process of rate \( n \), independent of \( X \), and let

\[
0 = S_0^n < S_1^n < S_2^n < \cdots < S_k^n < \cdots
\]

denote the jump times of \( \Lambda^n \). Define

\[
X^n_t = X_{S^n_k} \quad \text{for} \quad S^n_k \leq t < S^n_{k+1} \quad (k = 0, 1, 2, \ldots),
\]

and notice that

\[
Y^n(k) := X^n_{S^n_k} \quad k = 0, 1, 2, \ldots,
\]

is a realization of the Markov chain described before the statement of the Theorem. We can (and do) assume that the rate \( n \) Poisson process \( \Pi^n \) of the theorem is independent of \( X \) and of \( \Lambda^n \). Let

\[
0 = T^n_0 < T^n_1 < T^n_2 < \cdots < T^n_k < \cdots
\]

denote the jump times of \( \Pi^n \). Finally, let \( A^n(t) \) be the increasing piece-wise linear process that interpolates the points \((T^n_k, S^n_k), k = 0, 1, 2, \ldots\). From (3) and (6) we deduce that \( Z^n_t \) admits the representation

\[
Z^n_t = X^n_{A^n(t)}.
\]

The convergence in distribution of \( Z^n \) to \( X \) will follow from two observations:

(i) Because the “points” of \( \Lambda^n \) become dense (in probability) as \( n \to \infty \) and because \( X \) has rcll paths, \( X^n \) converges in probability to \( X \), the convergence being uniform on compact time intervals. That is,

\[
\lim_{n} \mathbb{P}^x \left[ \sup_{0 \leq s \leq t} d(X_s, X^n_s) > \epsilon \right] = 0 \quad \forall \epsilon > 0.
\]

(In fact, if we take (as we may) \( \Lambda^n \) to be \( \Lambda^{(1)} + \Lambda^{(2)} + \cdots + \Lambda^{(n)} \) where the \( \{\Lambda^{(k)}\}_{k \geq 1} \) is a sequence of independent unit-rate Poisson processes, then \( \sup_{0 \leq s \leq t} d(X_s, X^n_s) \to 0 \) almost surely as \( n \to \infty \).)
(ii) $A^n(t)$ converges to $t$ in probability as $n \to \infty$, and again the convergence is uniform on compacts:

$$
\lim_{n} \mathbb{P}^x \left[ \sup_{0 \leq s \leq t} |A^n(s) - s| > \epsilon \right] = 0 \quad \forall \epsilon > 0.
$$

To see this notice that the related process

$$
B^n(t) := S^n_k \text{ for } T^n_k \leq t < T^n_{k+1} \quad (k = 0, 1, 2, \ldots)
$$

is a subordinator [1] with Lévy exponent

$$
\varphi_n(\lambda) = -\frac{1}{t} \log \mathbb{P}^x(\exp(-\lambda B^n(t))) = n \int_0^\infty (1 - e^{-\lambda z})ne^{-nz} \, dz = \lambda \frac{n}{n + \lambda}.
$$

Thus, for fixed $t > 0$, $B^n(t)$ converges to $t$ in probability as $n \to \infty$. $t$. The monotonicity of $t \mapsto B^n(t)$ and the continuity of its limit allow us to conclude that the convergence is uniform on compacts (in probability). The asserted convergence of $A^n$ now follows by a simple bracketing argument.

Feeding (i) and (ii) into the representation (8) we see that, for each $t > 0$,

$$
\lim_{n} \mathbb{P}^x(d_{[0,t]}(Z^n, X) > \epsilon) = 0 \quad \forall \epsilon > 0,
$$

where $d_{[0,t]}$ is the Skorokhod distance in the path space $D([0,t], E)$; see [3]. In particular, $Z^n$ converges in distribution (in the Skorokhod topology) to $X$.

**References**


