**On the Approximation of Markov Processes**
**by Compound Poisson Processes**

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My intention in this note is to present a simple approach to the approximation of a continuous time Markov process by "compound Poisson" (or "pure jump") processes. Such approximations have been discussed recently in quite general contexts in [**5**] and [**4**]. Our setting is broader than that found in these references, and our proof more direct. A related, but much deeper, approximation of diffusion processes in Euclidean space is the subject of [**6**].

Let $X = (X_t, P^x)$ be a simple Markov process with right-continuous left-limited (rcll) sample paths, state space $E$, transition semigroup $(P_t)_{t\geq 0}$, and resolvent $(U^\alpha)_{\alpha>0}$. Thus, the action of $P_t$ on abounded Borel function $f: E \to \mathbf{R}$ is given by

$$(1) \qquad P_t f(x) = \mathbf{P}^x[f(X_t)], \qquad t \geq 0, x \in E,$$

where $\mathbf{P}^x$ is the law of $X$ under the initial condition $X_0 = x$; and

$$(2) \qquad U^\alpha f(x) := \mathbf{P}^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt\right] = \int_0^\infty e^{-\alpha t} P_t f(x)\, dt.$$

So as not to obscure the main idea, we assume for simplicity that $E$ is homeomorphic to a Borel subset of a compact metric space, and that $P_t f$ is Borel measurable whenever the bounded function $f$ is so. From this it follows that $U^\alpha$ also preserves Borel measurability.

For each positive integer $n$, the kernel $K_n(x, dy) := nU^n(x, dy)$ is a probability kernel on $E$; evidently $K_n(x, \cdot)$ is the $\mathbf{P}^x$-distribution of $X_T$, where $T$ is independent of $X$ and exponentially distributed with rate $n$. Let $(Y^n(k) : k = 0, 1, 2, \ldots)$ be a Markov chain on $E$ with one-step transition kernel $K_n$. Let $\Pi^n = (\Pi^n(t))_{t\geq 0}$ be a Poisson process of rate $n$, independent of $Y^n$, and define a continuous-time pure jump Markov process by

$$(3) \qquad Z_t^n := Y^n(\Pi^n(t)), \qquad t \geq 0.$$

The construction of such processes is discussed in detail in section of [**2**]. (In particlar, $Z^n$ is a strong Markov process.) The following result is proved in [**5**] for reversible Markov processes and in [**4**] for Hunt processes.

**Theorem.** *If $Y^n(0) = x$, then the sequence $\{Z^n\}_{n\geq 1}$ of rcll processes converges weakly to $X$ (under $\mathbf{P}^x$) as $n \to \infty$.*

*Proof.* Fix $x \in E$; in the following discussion, the law $\mathbf{P}^x$ will govern $X$. Let $\Lambda^n = (\Lambda^n(t))_{t\geq 0}$ be a Poisson process of rate $n$, independent of $X$, and let

$$0 = S_0^n < S_1^n < S_2^n < \cdots < S_k^n < \cdots \tag{4}$$

denote the jump times of $\Lambda^n$. Define

$$X_t^n = X_{S_k^n} \quad \text{for } S_k^n \leq t < S_{k+1}^n \quad (k = 0, 1, 2, \ldots), \tag{5}$$

and notice that

$$Y^n(k) := X^n_{S_k^n} \qquad k = 0, 1, 2, \ldots, \tag{6}$$

is a realization of the Markov chain described before the statement of the Theorem. We can (and do) assume that the rate $n$ Poisson process $\Pi^n$ of the theorem is independent of $X$ and of $\Lambda^n$. Let

$$0 = T_0^n < T_1^n < T_2^n < \cdots < T_k^n < \cdots \tag{7}$$

denote the jump times of $\Pi^n$. Finally, let $A^n(t)$ be the increasing piece-wise linear process that interpolates the points $(T_k^n, S_k^n)$, $k = 0, 1, 2, \ldots$. From (3) and (6) we deduce that $Z_t^n$ admits the representation

$$Z_t^n = X^n_{A^n(t)}. \tag{8}$$

The convergence in distribution of $Z^n$ to $X$ will follow from two observations:

(i) Because the "points" of $\Lambda^n$ become dense (in probability) as $n \to \infty$ and because $X$ has rcll paths, $X^n$ converges in probability to $X$, the convergence being uniform on compact time intervals. That is,

$$\lim_n \mathbf{P}^x \left[ \sup_{0\leq s\leq t} d(X_s, X_s^n) > \epsilon \right] = 0 \qquad \forall \epsilon > 0. \tag{9}$$

(In fact, if we take (as we may) $\Lambda^n$ to be $\Lambda^{(1)} + \Lambda^{(2)} + \cdots + \Lambda^{(n)}$ where the $\{\Lambda^{(k)}\}_{k\geq 1}$ is a sequence of independent unit-rate Poisson processes, then $\sup_{0\leq s\leq t} d(X_s, X_s^n) \to 0$ almost surely as $n \to \infty$.)

(ii) $A^n(t)$ converges to $t$ in probability as $n \to \infty$, and again the convergence is uniform on compacts:

$$\lim_n \mathbf{P}^x \left[ \sup_{0 \le s \le t} |A^n(s) - s| > \epsilon \right] = 0 \qquad \forall \epsilon > 0. \tag{10}$$

To see this notice that the related process

$$B^n(t) := S_k^n \quad \text{for } T_k^n \le t < T_{k+1}^n \quad (k = 0, 1, 2, \ldots) \tag{11}$$

is a subordinator [**1**] with Lévy exponent

$$\varphi_n(\lambda) = -\frac{1}{t} \log P^x(\exp(-\lambda B^n(t))) = n \int_0^\infty (1 - e^{-\lambda z}) n e^{-nz}\, dz = \lambda \frac{n}{n+\lambda}. \tag{12}$$

Thus, for fixed $t > 0$, $B^n(t)$ converges to $t$ in probability as $n \to \infty$. $t$. The monotonicity of $t \mapsto B^n(t)$ and the continuity of its limit allow us to conclude that the convergence is uniform on compacts (in probability). The asserted convergence of $A^n$ now follows by a simple bracketing argument.

Feeding (i) and (ii) into the representation (8) we see that, for each $t > 0$,

$$\lim_n P^x(d_{[0,t]}(Z^n, X) > \epsilon) = 0 \qquad \forall \epsilon > 0,$$

where $d_{[0,t]}$ is the Skorokhod distance in the path space $D([0,t], E)$; see [**3**]. In particular, $Z^n$ converges in distribution (in the Skorokhod topology) to $X$.

## References


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