Martingale Functions of Brownian Motion and its Local Time

by

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ABSTRACT

We characterize the class of local martingales of the form $H(B_t, L_t)$ for a standard one-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ and its local time at 0, $L = (L_t)_{t\geq 0}$. The main result is closely related to work of J. Obłój, who studied the local martingales of the form $H(B_t, \overline{B}_t)$, where $\overline{B}_t = \sup_{0\leq s\leq t} B_s$.

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1. Results.

Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion with $B_0 = 0$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by B (augmented by the $\mathbf{P}$-null sets in $\sigma\{B_t : t \geq 0\}$). The local time process $L = (L_t)_{t\geq 0}$ measures the zero set $\mathcal{Z} := \{t \geq 0 : B_t = 0\}$. The process L is adapted to $(\mathcal{F}_t)$, continuous, increasing, and flat on the complement of $\mathcal{Z}$; and can be constructed as the almost sure limit

$$L_t = \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|B_s|<\epsilon\}}\, ds, \tag{1.1}$$

the convergence being uniform on compact time intervals. Alternatively, L is characterized by Tanaka's formula:

$$L_t = |B_t| - \int_0^t \operatorname{sgn}(B_s)\, dB_s, \qquad \forall t \geq 0, \qquad \text{a.s.} \tag{1.2}$$

The local time also features in the one-sided variants of Tanaka's formula; the "positive" case being

$$B_t^+ = \int_0^t 1_{\{B_s>0\}}\, dB_s + \frac{1}{2} L_t, \qquad \forall t \geq 0, \qquad \text{a.s.,} \tag{1.3}$$

where $b^+ := b \vee 0$ for a real number b.

If $f : [0, \infty[\to \mathbf{R}$ is continuously differentiable, then (1.3) and Itô's formula yield

$$B_t^+ \cdot f(L_t) = \int_0^t f(L_s) \cdot 1_{\{B_s>0\}}\, dB_s + \frac{1}{2} F(L_t) \tag{1.4}$$

where $F(x) := \int_0^x f(u)\, du$, $x \geq 0$. A monotone class argument shows that (1.4) persists for general bounded measurable f, and then a truncation argument based on the following result yields the validity of (1.4) when f is merely locally integrable on $[0, \infty[$ (a condition that is clearly necessary as well).

(1.5) Lemma. *Let $f : [0, \infty[\to \mathbf{R}$ be a measurable function. Then*

$$\int_0^t [f(L_s)]^2\, ds < \infty \quad \forall t \geq 0, \text{ a.s.} \quad \Longleftrightarrow \quad \int_0^x |f(u)|\, du < \infty, \forall x > 0. \tag{1.6}$$

Proof. A theorem of Lévy tells us that

$$\left(\overline{B}_t, \overline{B}_t - B_t\right)_{t\geq 0} \stackrel{d}{=} \left(L_t, |B_t|\right)_{t\geq 0}, \tag{1.7}$$

so the assertion of the lemma follows from [**OY**; p. 523]; see also [**O**; p. 962]. ∎

Combining the above discussion with the analogous remarks concerning $B_t^- := (-B_t) \vee 0$, we arrive at the following result.

(1.8) Theorem. *Fix $f_+,\, f_- \in L^1_{\rm loc}[0,\infty[$ and define*

$$M_t := H(B_t, L_t), \qquad t \ge 0, \tag{1.9}$$

where

$$H(x,y) := x^+ \cdot f_+(y) - x^- \cdot f_-(y) - \frac{1}{2}\left[F_+(y) - F_-(y)\right], \tag{1.10}$$

and

$$F_\pm(y) := \int_0^y f_\pm(u)\, du, \qquad y \ge 0. \tag{1.11}$$

Then $M = (M_t)_{t\ge 0}$ is a continuous $(\mathcal{F}_t)_{t\ge 0}$ local martingale, with stochastic integral representation

$$M_t = \int_0^t \left[f_+(L_s)\cdot 1_{\{B_s>0\}} + f_-(L_s)\cdot 1_{\{B_s<0\}}\right] dB_s, \tag{1.12}$$

for all $t \ge 0$, almost surely.

Local martingales of the above type find application in [**AY**] and [**RVY**]. Two special cases are worth noting: $f_+ = f_-$ and $f_+ = -f_-$.

(1.13) Corollary. *Let f be in $L^1_{\rm loc}$ and define $F(x) := \int_0^x f(u)\, du$ as before.*

(a) The process $B_t \cdot f(L_t)$, $t \ge 0$, is a continuous local martingale, and

$$B_t \cdot f(L_t) = \int_0^t f(L_s)\, dB_s, \qquad \forall t \ge 0, \text{a.s.} \tag{1.14}$$

(b) The process $|B_t| \cdot f(L_t) - F(L_t)$, $t \ge 0$, is a continuous local martingale, and

$$|B_t| \cdot f(L_t) - F(L_t) = \int_0^t f(L_s)\cdot \operatorname{sgn}(B_s)\, dB_s, \qquad \forall t \ge 0, \text{a.s.} \tag{1.15}$$

Part (a) of Corollary (1.13) is a special case of "balayage" considerations found in [**Y**; Thm. 2].

The following converse of Theorem (1.8) is the main result of this paper. It complements and (as a consequence of Lévy's Theorem mentioned previously) generalizes Obłój's characterization of the local martingale functions of Brownian motion and its running maximum. A footnote on p. 958 of [**O**] indicates that Obłój has obtained a similar result by the methods of [**O**]. The proof presented below is a condensed version of an argument found in the second-named author's Ph. D. thesis [**W**], and it follows the broad outlines of the approach used in [**O**].

(1.16) Theorem. *Let $H : \mathbf{R}\times[0,\infty[\to \mathbf{R}$ be a Borel measurable function, with $H(0,0) = 0$, such that $(H(B_t, L_t))_{t\ge 0}$ is a continuous local martingale with respect to $(\mathcal{F}_t)_{t\ge 0}$. Then there exist locally integrable functions f_+ and f_- such that, if we define $F_\pm(y) = \int_0^y f_\pm(u)\, du$ and*

$$\tilde H(x,y) := x^+ \cdot f_+(y) - x^- \cdot f_-(y) - \frac{1}{2}\left[F_+(y) - F_-(y)\right], \tag{1.17}$$

then

$$H(B_t, L_t) = \tilde H(B_t, L_t), \qquad \forall t \ge 0, \ \text{a.s.} \tag{1.18}$$

For the following corollary recall that $\mathcal{Z} := \{t \ge 0 : B_t = 0\}$.

(1.19) Corollary. *Let* $H : \mathbf{R} \times [0,\infty[\to \mathbf{R}$ *be a measurable function, with* $H(0,0) = 0$*, such that* $M_t := H(B_t, L_t)$*,* $t \geq 0$*, is a continuous local martingale with respect to* $(\mathcal{F}_t)_{t \geq 0}$*. Suppose further that*

$$M_t = 0, \quad \forall t \in \mathcal{Z}, \tag{1.20}$$

almost surely. Then

$$M_t = B_t \cdot f(L_t), \qquad \forall t \geq 0, \text{ a.s.}, \tag{1.21}$$

for some $f \in L^1_{\text{loc}}$*.*

(1.22) Remarks. (a) A direct argument using Itô's formula shows that if H is of the class $C^{2,1}$ with $H(0,0) = 0$, and if $M_t := H(B_t, L_t)$ is a (continuous) local martingale, then M must be of the form (1.21); cf. [**OY**; pp. 522–523]. Notice that (1.16) permits us to make the same deduction under the weaker smoothness condition that $x \mapsto H(x, y)$ is differentiable at $x = 0$ for Lebesgue a.e. $y \geq 0$.

(b) The new results presented in this paper are taken from the second-named author's doctoral dissertation [**W**].

(c) The pair (B, L) is a strong Markov process on the state space $E := \mathbf{R} \times [0, \infty[$. Under the law $\mathbf{P}^x$ of Brownian motion started at $x \in \mathbf{R}$, the process $(B_t, y + L_t)_{t \geq 0}$ is a realization of this process with starting point $(x, y) \in E$. Let us say that a Borel function $H : E \to \mathbf{R}$ is *harmonic* provided $M_t := H(B_t, y + L_t)$, $t \geq 0$, is a continuous $\mathbf{P}^x$-local martingale for each starting point $(x, y) \in E$. It is not hard to deduce from (1.8) that if H is given by the right side of (1.17), then H is harmonic. Conversely, if H is harmonic, then there are locally integrable functions f_+ and f_- such that $H(x, y)$ is equal to the right side of (1.17) for *all* $(x, y) \in E$. We sketch the proof. If we start at (x, y) with $x > 0$, then $M_t = H(B_t, y)$ for $0 \leq t \leq T_0$, where $T_z := \inf\{t : B_t = z\}$ for $z \in \mathbf{R}$. As noted already in (b), the harmonic functions of Brownian motion on $]0, \infty[$ with absorbing barrier at 0 are the affine functions. Thus there are Borel functions φ_+ and γ_+ mapping $[0, \infty[$ to $\mathbf{R}$ such that for each $x > 0$ and $y \geq 0$,

$$M_t = H(B_t, y + L_t) = B_t \cdot \varphi_+(y) + \gamma_+(y), \qquad 0 \leq t \leq T_0, \tag{1.26}$$

$\mathbf{P}^x$-a.s. Fix $x > 0$, start the process (B, L) at $(0, 0)$, and consider the local martingale M on the time interval $[T_x, D_x]$, where $D_x := \inf\{t > T_x : B_t = 0\}$. The strong Markov property at time T_x implies that

$$M_t = B_t \cdot \varphi_+(L_t) + \gamma_+(L_t), \qquad T_x \leq t \leq D_x, \tag{1.27}$$

$\mathbf{P}^0$-a.s. Contrasting this with the information provided by Theorem (1.16), we deduce that

$$\varphi_+(y) = f_+(y) \tag{1.28}$$

for Lebesgue a.e. $y \geq 0$, because the $\mathbf{P}^0$-distribution of L_{T_x} has a strictly positive density with respect to Lebesgue measure. Thus, substituting φ_+ for f_+ we can suppose that, in addition to the conclusion of Theorem (1.16) holding, we have the identity

$$H(x,y) = x \cdot f_+(y) + H(0,y), \qquad \forall (x,y) \in]0,\infty[\times[0,\infty[. \tag{1.26}$$

Similarly,

$$H(x,y) = x \cdot f_-(y) + H(0,y), \qquad \forall (x,y) \in]-\infty,0[\times[0,\infty[. \tag{1.27}$$

Together (1.26) and (1.27) yield

$$H(x,y) = x^+ \cdot f_+(y) - x^- \cdot f_-(y) + H(0,y), \qquad \forall (x,y) \in \mathbf{R}\times[0,\infty[. \tag{1.28}$$

To verify the asserted form of $H(0,y)$, note that (1.28), the local martingale property of $H(B_t,L_t)$, and Theorem (1.8) imply that

$$2H(0,L_t) + F_+(L_t) - F_-(L_t), \qquad t \geq 0, \tag{1.29}$$

is a continuous $\mathbf{P}^0$-local martingale. This is enough to conclude that $H(0,y) = [F_-(y) - F_+(y)]/2$ for all $y \geq 0$—see the discussion following (2.20) in the proof of Theorem (1.16) found in section 2.

(d) When the function H in Theorem (1.16) is symmetric in its first argument (i.e., when $H(x,y) = G(|x|,y)$ for some G), then we recover the main result of [**O**] because of (1.7)—the distribution of $(|B_t|, L_t)_{t\geq 0}$ is that of a reflecting Brownian motion on $[0,\infty[$ paired with its local time at 0. We record this result for the sake of completeness.

(1.30) Theorem. [**O**; Thm. 2] *Let $(R_t)_{t\geq 0}$ be a reflecting Brownian motion on $[0,\infty[$ with $R_0 = 0$, and let $(K_t)_{t\geq 0}$ be its (semimartingale) local time at 0. Let $\Phi : \mathbf{R}\times[0,\infty[$ be a Borel measurable function such that $\Phi(R_t,K_t)$, $t \geq 0$, is a continuous local martingale (relative to its natural filtration). Then there is a locally integrable function $g : [0,\infty[$ such that*

$$\begin{aligned} \Phi(R_t,K_t) - \Phi(0,0) &= R_t \cdot g(K_t) - G(K_t), \\ &= \int_0^t g(K_s)\, dR_s - \int_0^t g(K_s)\, dK_s, \end{aligned} \tag{1.31}$$

for all $t \geq 0$, almost surely, where $G(x) := \int_0^x g(u)\, du$ for $x \geq 0$.

2. Proofs.

Proof of Theorem (1.16).

The proof is close in outline to Obłój's proof of [**O**; Thm. 1], so certain points will be only sketched.

Let $H : [0,\mathbf{R}\times[0,\infty[\to \mathbf{R}$ be a Borel function such that $(H(B_t,L_t))_{t\geq 0}$ is a continuous local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Define $M_t := H(B_t,L_t) - H(B_0,L_0)$. By the

Brownian Local Martingale Representation Theorem, there exists a real-valued predictable process η such that

$$M_t = \int_0^t \eta_s \, dB_s, \qquad \forall t \geq 0, \quad \text{a.s.} \tag{2.1}$$

Fix $0 < s < t$ and observe that the integrand η_s is equal to

$$\lim_{v \downarrow s} \frac{\partial}{\partial v} \mathbf{E}\left[H(B_t, L_t) B_v \,|\, \mathcal{F}_s\right],$$

provided $(H(B_t, L_t))_{t \geq 0}$ is a square integrable martingale. Since the pair (B_t, L_t) is a Markov process, this makes it intuitively plausible that η takes the form

$$\eta_s = h(B_s, L_s), \qquad \forall s \geq 0 \quad \text{a.s.} \tag{2.2}$$

for some Borel function h; see pp. 963–964 in [**O**] for a more detailed discussion of this point. Summarizing, we now have the following two identities:

$$M_t = H(B_t, L_t) - H(B_0, L_0), \tag{2.3}$$

and

$$M_t = \int_0^t h(B_s, L_s) \, dB_s. \tag{2.4}$$

Define stopping times, for fixed x, by

$$T_x := \inf\{t > 0 : B_t = x\}, \quad U_x := \inf\{t > T_x : B_t = 0\},$$

and set

$$\xi_x := L_{T_x}, \qquad R_x := U_x - T_x.$$

Also, for $s \geq 0$, define

$$\beta_s = B_{T_x+s} - B_{T_x}, \quad D_s = \inf\{t > s : B_t = 0\}.$$

Note that $t \mapsto L_t$ is constant on $[T_x, T_x + R_x]$, and that ξ_x (being $\mathcal{F}_{T_x}$-measurable) is independent of $(\beta_s)_{s \geq 0}$. We sometimes write $\beta_s^x := B_{T_x+s} = x + \beta_s$, a Brownian motion started at x. By expanding

$$M_{T_x+(t \wedge R_x)} - M_{T_x}$$

in two different ways using (2.3) and (2.4) we obtain

$$\int_0^{t \wedge R_x} h(x + \beta_u, \xi_x) \, d\beta_u = H(x + \beta_{t \wedge R_x}, \xi_x) - H(x, \xi_x) \quad \forall t \geq 0 \text{ a.s.}. \tag{2.5}$$

The stochastic integral in (2.5) is a local martingale, hence so is the right side. However this is a local martingale function of (absorbed) Brownian motion and thus is an affine function of that Brownian motion. Hence, almost surely for all $t \in [0, R_x]$,

$$H(x+\beta_t, \xi_x) - H(x, \xi_x) = f(x, \xi_x) \cdot \beta_t + g(x, \xi_x) = \int_0^t f(x, \xi_x)\, d\beta_u + g(x, \xi_x), \tag{2.6}$$

for certain functions f and g. Taking $t = 0$ we find that $g(x, \xi_x)$ is identically 0. Thus, almost surely for all $t \in [0, R_x]$,

$$H(x+\beta_t, \xi_x) - H(x, \xi_x) = \int_0^t f(x, \xi_x)\, d\beta_u. \tag{2.7}$$

We now verify that $f(x, y)$ does not depend on x. To this end fix $x \neq 0$ and note that, by the independence of β and ξ_x, and the fact that ξ_x has a density with respect to Lebesgue measure, we obtain

$$\int_0^{t\wedge R_x} h(x+\beta_u, y)\, d\beta_u = \int_0^{t\wedge R_x} f(x, y)\, d\beta_u, \tag{2.8}$$

for a.e. $y \in [0, \infty)$ with respect to Lebesgue measure. Now suppose that $x > 0$. Fix x' such that $0 < x' < x$ and define the following:

$$T^{'} := \inf(u > T_x : B_u = x'), \qquad \beta^{'}_s := B_{T'+s}.$$

Exploiting the fact that the β' path is a fragment of the β path:

$$\beta^{'}_s = \beta^x_{s+R},$$

where $R := (T^{'} - T_x)$, we conclude that for $0 \le t \le (U_x - T^{'})$,

$$\int_0^t f(x', y)\, d\beta^{'}_u = \int_0^t f(x, y)\, d\beta^{'}_u, \tag{2.9}$$

which implies that for $t \in [0, U_x - T']$ there exists a Lebesgue null set N_x such that for all $y \in [0, \infty) \setminus N_x$ we have

$$f(x', y) = f(x, y), \qquad \forall x' \in]0, x[. \tag{2.10}$$

We now let x vary through $\mathbf{N}$ to conclude that there exists a single function $f_+ : [0, \infty) \to \mathbf{R}$, uniquely determined on $[0, \infty) \setminus \bigcup_{n\in\mathbf{N}} N_n$, such that

$$H(x+\beta_t, \xi_x) - H(x, \xi_x) = \int_0^t f_+(\xi_x)\, d\beta_u = f_+(\xi_x) \cdot \beta_t, \tag{2.11}$$

for $0 \le t \le R_x$, a.s. A similar argument holds for $x < 0$ and we thus obtain for $t \in [0, R_x]$,

$$H(x+\beta_t, L_{T_x}) - H(x, L_{T_x}) = \begin{cases} \int_0^t f_+(L_{T_x})\, d\beta_u, & \text{for } x = \beta_0^x > 0, \\ \int_0^t f_-(L_{T_x})\, d\beta_u, & \text{for } x = \beta_0^x < 0, \end{cases} \tag{2.12}$$

where we have denoted by f_+, and f_- the functions corresponding to excursions above and below 0 respectively, and $\xi_x = L_{T_x}$.

The above considerations have involved the *first* excursion of B (away from 0) to reach the level x, but they apply to subsequent excursions as well. A little thought now shows that (2.12) implies (via the Markov property and the fact that the local time is constant over each excursion) the following: For each fixed $r \ge 0$, almost surely on $\{B_r \neq 0\}$,

$$H(B_s, L_s) - H(B_r, L_r) = \begin{cases} f_+(L_s)(B_s - B_r), & \text{for } B_r > 0, \\ f_-(L_s)(B_s - B_r), & \text{for } B_r < 0, \end{cases} \tag{2.13}$$

for all $s \in [r, D_r]$, where D_r is as before: $D_r = \inf\{t > r : B_t = 0\}$. By (2.13) and an approximation argument we may then conclude that

$$H(B_t, L_t) - H(0, L_t) = B_t^+ \cdot f_+(L_t) - B_t^- \cdot f_-(L_t), \quad \forall t \ge 0, \text{ a.s.} \tag{2.14}$$

By (2.4) and the above discussion, we have computed the Itô integrand η;

$$\eta_s = f_+(L_s) \cdot 1_{\{B_s > 0\}} + f_-(L_s) \cdot 1_{\{B_s < 0\}} \text{ for a.e. pair } (s, \omega). \tag{2.15}$$

In view of (1.1) the local time L enjoys the following symmetry property:

$$(B_\cdot, L_\cdot) \overset{d}{=} (-B_\cdot, L_\cdot)\,. \tag{2.16}$$

From (2.16), (1.5), and the fact that $(H(B_t, L_t))_{t \ge 0}$ is a continuous local martingale we now deduce that both f_+, and f_- are in L^1_{loc}. This in turn allows us to apply (1.8) to conclude that

$$B_t^+ \cdot f_+(L_t) = \int_0^t f_+(L_s) \cdot 1_{\{B_s > 0\}}\, dB_s + \frac{1}{2} F_+(L_t) \tag{2.17}$$

and

$$B_t^- \cdot f_-(L_t) = -\int_0^t f_-(L_s) \cdot 1_{\{B_s < 0\}}\, dB_s + \frac{1}{2} F_-(L_t), \tag{2.18}$$

where $F_\pm(y) = \int_0^y f_\pm(u)\, du$. Therefore the following holds,

$$\begin{aligned} H(B_t, L_t) - H(0, L_t) &= f_+(L_t) \cdot B_t^+ - f_-(L_t) \cdot B_t^-, \\ &= \int_0^t f_+(L_s) \cdot 1_{\{B_s > 0\}}\, dB_s + \int_0^t f_-(L_s) \cdot 1_{\{B_s < 0\}}\, dB_s \\ &\quad + \frac{1}{2} \left(F_+(L_t) - F_-(L_t) \right) \end{aligned} \tag{2.19}$$

for all $t \geq 0$, almost surely. This implies that

$$N_t := H(0, L_t) + \frac{1}{2}\left[F_+(L_t) - F_-(L_t)\right], \qquad t \geq 0, \tag{2.20}$$

is a continuous local martingale. Since the local time L is flat off the zero set of B, the local martingale N must have vanishing quadratic variation, hence it must be constant in time. Consequently,

$$-H(0, L_t) = \frac{1}{2}\left[F_+(L_t) - F_-(L_t)\right], \qquad \forall t \geq 0, \text{ a.s.}, \tag{2.21}$$

yielding the main assertion in Theorem (1.16). □

Proof of Corollary (1.19).

Let $M_t = H(B_t, L_t)$ be a continuous local martingale vanishing on the zero set $\mathcal{Z}$ of B. By Theorem (1.16) we have

$$0 = F_+(L_t) - F_-(L_t), \qquad t \in \mathcal{Z}, \tag{2.22}$$

almost surely. But as t increases through $\mathcal{Z}$, the local time process L increases continuously from 0 to $+\infty$. In other words, $L(\mathcal{Z}) = [0, \infty[$, almost surely. It follows from this and (2.22) that $F_+(x) = F_-(x)$ for all $x \geq 0$. Consequently, $f_+ = f_-$ Lebesgue a.e. on $[0, \infty[$. Define $f := f_+$. Since the distribution of L_t has a density with respect to Lebesgue measure, we have $f(L_t) = f_+(L_t) = f_-(L_t)$ almost surely, for each fixed $t > 0$. Feeding this into (1.17) and (1.18) we deduce that

$$H(B_t, L_t) = B_t \cdot f(L_t), \tag{2.23}$$

first for each fixed $t > 0$, and then almost surely for all $t \geq 0$ since both sides of (2.23) are continuous in t. □

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