Martingale Functions of Brownian Motion and its Local Time

by

P.J. Fitzsimmons and D.M. Wroblewski

Department of Mathematics, 0112
University of California San Diego
9500 Gilman Drive
La Jolla, CA 92093–0112
USA

September 25, 2008

ABSTRACT

We characterize the class of local martingales of the form $H(B_t, L_t)$ for a standard one-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ and its local time at 0, $L = (L_t)_{t\geq 0}$. The main result is closely related to work of J. Obłój, who studied the local martingales of the form $H(B_t, \overline{B}_t)$, where $\overline{B}_t = \sup_{0\leq s\leq t} B_s$.

Keywords. Brownian motion, local martingale, local time.
2000 Mathematics Subject Classification. Primary: 60J65, Secondary: 60J55, 60G44.

1. Results.

Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion with $B_0 = 0$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by B (augmented by the $\mathbf{P}$-null sets in $\sigma\{B_t : t \geq 0\}$). The local time process $L = (L_t)_{t\geq 0}$ measures the zero set $\mathcal{Z} := \{t \geq 0 : B_t = 0\}$. The process L is adapted to $(\mathcal{F}_t)$, continuous, increasing, and flat on the complement of $\mathcal{Z}$; and can be constructed as the almost sure limit

$$L_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|B_s|<\epsilon\}}\, ds, \tag{1.1}$$

the convergence being uniform on compact time intervals. Alternatively, L is characterized by Tanaka's formula:

$$L_t = |B_t| - \int_0^t \operatorname{sgn}(B_s)\, dB_s, \qquad \forall t \geq 0, \qquad \text{a.s.} \tag{1.2}$$

The local time also features in the one-sided variants of Tanaka's formula; the "positive" case being

$$B_t^+ = \int_0^t 1_{\{B_s>0\}}\, dB_s + \frac{1}{2} L_t, \qquad \forall t \geq 0, \qquad \text{a.s.}, \tag{1.3}$$

where $b^+ := b \vee 0$ for a real number b.

If $f : [0, \infty[\to \mathbf{R}$ is continuously differentiable, then (1.3) and Itô's formula yield

$$B_t^+ \cdot f(L_t) = \int_0^t f(L_s) \cdot 1_{\{B_s>0\}}\, dB_s + \frac{1}{2} F(L_t) \tag{1.4}$$

where $F(x) := \int_0^x f(u)\, du$, $x \geq 0$. A monotone class argument shows that (1.4) persists for general bounded measurable f, and then a truncation argument based on the following result yields the validity of (1.4) when f is merely locally integrable on $[0, \infty[$ (a condition that is clearly necessary as well).

(1.5) Lemma. *Let $f : [0, \infty[\to \mathbf{R}$ be a measurable function. Then*

$$\int_0^t [f(L_s)]^2\, ds < \infty \quad \forall t \geq 0, \text{ a.s.} \quad \Longleftrightarrow \quad \int_0^x |f(u)|\, du < \infty, \forall x > 0. \tag{1.6}$$

Proof. A theorem of Lévy tells us that

$$\left(\overline{B}_t, \overline{B}_t - B_t\right)_{t\geq 0} \stackrel{d}{=} \left(L_t, |B_t|\right)_{t\geq 0}, \tag{1.7}$$

so the assertion of the lemma follows from [**OY**; p. 523]; see also [**O**; p. 962]. ∎

Combining the above discussion with the analogous remarks concerning $B_t^- := (-B_t) \vee 0$, we arrive at the following result.

(1.8) Theorem. *Fix* $f_+,\, f_- \in L^1_{\mathrm{loc}}[0,\infty[$ *and define*

$$M_t := H(B_t, L_t), \qquad t \geq 0, \tag{1.9}$$

where

$$H(x,y) := x^+ \cdot f_+(y) - x^- \cdot f_-(y) - \frac{1}{2}\left[F_+(y) - F_-(y)\right], \tag{1.10}$$

and

$$F_\pm(y) := \int_0^y f_\pm(u)\, du, \qquad y \geq 0. \tag{1.11}$$

Then $M = (M_t)_{t\geq 0}$ *is a continuous* $(\mathcal{F}_t)_{t\geq 0}$ *local martingale, with stochastic integral representation*

$$M_t = \int_0^t \left[f_+(L_s)\cdot 1_{\{B_s>0\}} + f_-(L_s)\cdot 1_{\{B_s<0\}}\right] dB_s, \tag{1.12}$$

for all $t \geq 0$*, almost surely.*

Local martingales of the above type find application in [**AY**] and [**RVY**]. Two special cases are worth noting: $f_+ = f_-$ and $f_+ = -f_-$.

(1.13) Corollary. *Let* f *be in* L^1_{loc} *and define* $F(x) := \int_0^x f(u)\, du$ *as before.*

(a) The process $B_t \cdot f(L_t)$*,* $t \geq 0$*, is a continuous local martingale, and*

$$B_t \cdot f(L_t) = \int_0^t f(L_s)\, dB_s, \qquad \forall t \geq 0, \text{a.s.} \tag{1.14}$$

(b) The process $|B_t| \cdot f(L_t) - F(L_t)$*,* $t \geq 0$*, is a continuous local martingale, and*

$$|B_t| \cdot f(L_t) - F(L_t) = \int_0^t f(L_s)\cdot \mathrm{sgn}(B_s)\, dB_s, \qquad \forall t \geq 0, \text{a.s.} \tag{1.15}$$

Part (a) of Corollary (1.13) is a special case of "balayage" considerations found in [**Y**; Thm. 2].

The following converse of Theorem (1.8) is the main result of this paper. It complements and (as a consequence of Lévy's Theorem mentioned previously) generalizes Obłój's characterization of the local martingale functions of Brownian motion and its running maximum. A footnote on p. 958 of [**O**] indicates that Obłój has obtained a similar result by the methods of [**O**]. The proof presented below is a condensed version of an argument found in the second-named author's Ph. D. thesis [**W**], and it follows the broad outlines of the approach used in [**O**].

(1.16) Theorem. *Let* $H : \mathbf{R} \times [0,\infty[\to \mathbf{R}$ *be a Borel measurable function, with* $H(0,0) = 0$*, such that* $(H(B_t, L_t))_{t\geq 0}$ *is a continuous local martingale with respect to* $(\mathcal{F}_t)_{t\geq 0}$*. Then there exist locally integrable functions* f_+ *and* f_- *such that, if we define* $F_\pm(y) = \int_0^y f_\pm(u)\, du$ *and*

$$\tilde{H}(x,y) := x^+ \cdot f_+(y) - x^- \cdot f_-(y) - \frac{1}{2}\left[F_+(y) - F_-(y)\right], \tag{1.17}$$

then

$$H(B_t, L_t) = \tilde{H}(B_t, L_t), \qquad \forall t \geq 0, \text{ a.s.} \tag{1.18}$$

For the following corollary recall that $\mathcal{Z} := \{t \geq 0 : B_t = 0\}$.

(1.19) Corollary. *Let $H : \mathbf{R} \times [0, \infty[\to \mathbf{R}$ be a measurable function, with $H(0,0) = 0$, such that $M_t := H(B_t, L_t)$, $t \geq 0$, is a continuous local martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$. Suppose further that*

$$M_t = 0, \quad \forall t \in \mathcal{Z}, \tag{1.20}$$

almost surely. Then

$$M_t = B_t \cdot f(L_t), \qquad \forall t \geq 0, \text{ a.s.}, \tag{1.21}$$

for some $f \in L^1_{\text{loc}}$.

(1.22) Remarks. (a) A direct argument using Itô's formula shows that if H is of the class $C^{2,1}$ with $H(0,0) = 0$, and if $M_t := H(B_t, L_t)$ is a (continuous) local martingale, then M must be of the form (1.21); cf. [**OY**; pp. 522–523]. Notice that (1.16) permits us to make the same deduction under the weaker smoothness condition that $x \mapsto H(x,y)$ is differentiable at $x = 0$ for Lebesgue a.e. $y \geq 0$.

(b) The new results presented in this paper are taken from the second-named author's doctoral dissertation [**W**].

(c) The pair (B, L) is a strong Markov process on the state space $E := \mathbf{R} \times [0, \infty[$. Under the law $\mathbf{P}^x$ of Brownian motion started at $x \in \mathbf{R}$, the process $(B_t, y + L_t)_{t \geq 0}$ is a realization of this process with starting point $(x, y) \in E$. Let us say that a Borel function $H : E \to \mathbf{R}$ is *harmonic* provided $M_t := H(B_t, y + L_t)$, $t \geq 0$, is a continuous $\mathbf{P}^x$-local martingale for each starting point $(x, y) \in E$. It is not hard to deduce from (1.8) that if H is given by the right side of (1.17), then H is harmonic. Conversely, if H is harmonic, then there are locally integrable functions f_+ and f_- such that $H(x,y)$ is equal to the right side of (1.17) for *all* $(x,y) \in E$. We sketch the proof. If we start at (x,y) with $x > 0$, then $M_t = H(B_t, y)$ for $0 \leq t \leq T_0$, where $T_z := \inf\{t : B_t = z\}$ for $z \in \mathbf{R}$. As noted already in (b), the harmonic functions of Brownian motion on $]0, \infty[$ with absorbing barrier at 0 are the affine functions. Thus there are Borel functions φ_+ and γ_+ mapping $[0, \infty[$ to $\mathbf{R}$ such that for each $x > 0$ and $y \geq 0$,

$$M_t = H(B_t, y + L_t) = B_t \cdot \varphi_+(y) + \gamma_+(y), \qquad 0 \leq t \leq T_0, \tag{1.26}$$

$\mathbf{P}^x$-a.s. Fix $x > 0$, start the process (B, L) at $(0,0)$, and consider the local martingale M on the time interval $[T_x, D_x]$, where $D_x := \inf\{t > T_x : B_t = 0\}$. The strong Markov property at time T_x implies that

$$M_t = B_t \cdot \varphi_+(L_t) + \gamma_+(L_t), \qquad T_x \leq t \leq D_x, \tag{1.27}$$

$\mathbf{P}^0$-a.s. Contrasting this with the information provided by Theorem (1.16), we deduce that

$$\varphi_+(y) = f_+(y) \tag{1.28}$$

for Lebesgue a.e. $y \geq 0$, because the $\mathbf{P}^0$-distribution of L_{T_x} has a strictly positive density with respect to Lebesgue measure. Thus, substituting φ_+ for f_+ we can suppose that, in addition to the conclusion of Theorem (1.16) holding, we have the identity

$$H(x,y) = x \cdot f_+(y) + H(0,y), \qquad \forall (x,y) \in]0,\infty[\times[0,\infty[. \tag{1.26}$$

Similarly,

$$H(x,y) = x \cdot f_-(y) + H(0,y), \qquad \forall (x,y) \in]-\infty,0[\times[0,\infty[. \tag{1.27}$$

Together (1.26) and (1.27) yield

$$H(x,y) = x^+ \cdot f_+(y) - x^- \cdot f_-(y) + H(0,y), \qquad \forall (x,y) \in \mathbf{R} \times [0,\infty[. \tag{1.28}$$

To verify the asserted form of $H(0,y)$, note that (1.28), the local martingale property of $H(B_t, L_t)$, and Theorem (1.8) imply that

$$2H(0,L_t) + F_+(L_t) - F_-(L_t), \qquad t \geq 0, \tag{1.29}$$

is a continuous $\mathbf{P}^0$-local martingale. This is enough to conclude that $H(0,y) = [F_-(y) - F_+(y)]/2$ for all $y \geq 0$—see the discussion following (2.20) in the proof of Theorem (1.16) found in section 2.

(d) When the function H in Theorem (1.16) is symmetric in its first argument (i.e., when $H(x,y) = G(|x|,y)$ for some G), then we recover the main result of [**O**] because of (1.7)—the distribution of $(|B_t|, L_t)_{t\geq 0}$ is that of a reflecting Brownian motion on $[0,\infty[$ paired with its local time at 0. We record this result for the sake of completeness.

(1.30) Theorem. [**O**; Thm. 2] *Let $(R_t)_{t\geq 0}$ be a reflecting Brownian motion on $[0,\infty[$ with $R_0 = 0$, and let $(K_t)_{t\geq 0}$ be its (semimartingale) local time at 0. Let $\Phi : \mathbf{R} \times [0,\infty[$ be a Borel measurable function such that $\Phi(R_t, K_t)$, $t \geq 0$, is a continuous local martingale (relative to its natural filtration). Then there is a locally integrable function $g : [0,\infty[$ such that*

$$\begin{aligned} \Phi(R_t,K_t) - \Phi(0,0) &= R_t \cdot g(K_t) - G(K_t), \\ &= \int_0^t g(K_s)\, dR_s - \int_0^t g(K_s)\, dK_s, \end{aligned} \tag{1.31}$$

for all $t \geq 0$, almost surely, where $G(x) := \int_0^x g(u)\, du$ for $x \geq 0$.

2. Proofs.

Proof of Theorem (1.16).

The proof is close in outline to Obłój's proof of [**O**; Thm. 1], so certain points will be only sketched.

Let $H : [0, \mathbf{R} \times [0,\infty[\to \mathbf{R}$ be a Borel function such that $(H(B_t,L_t))_{t\geq 0}$ is a continuous local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Define $M_t := H(B_t,L_t) - H(B_0,L_0)$. By the

Brownian Local Martingale Representation Theorem, there exists a real-valued predictable process η such that

$$M_t = \int_0^t \eta_s \, dB_s, \qquad \forall t \geq 0, \quad \text{a.s.} \tag{2.1}$$

Fix $0 < s < t$ and observe that the integrand η_s is equal to

$$\lim_{v \downarrow s} \frac{\partial}{\partial v} \mathbf{E}\left[H(B_t, L_t) B_v \,|\, \mathcal{F}_s \right],$$

provided $(H(B_t, L_t))_{t \geq 0}$ is a square integrable martingale. Since the pair (B_t, L_t) is a Markov process, this makes it intuitively plausible that η takes the form

$$\eta_s = h(B_s, L_s), \qquad \forall s \geq 0 \quad \text{a.s.} \tag{2.2}$$

for some Borel function h; see pp. 963–964 in [**O**] for a more detailed discussion of this point. Summarizing, we now have the following two identities:

$$M_t = H(B_t, L_t) - H(B_0, L_0), \tag{2.3}$$

and

$$M_t = \int_0^t h(B_s, L_s) \, dB_s. \tag{2.4}$$

Define stopping times, for fixed x, by

$$T_x := \inf\{t > 0 : B_t = x\}, \quad U_x := \inf\{t > T_x : B_t = 0\},$$

and set

$$\xi_x := L_{T_x}, \qquad R_x := U_x - T_x.$$

Also, for $s \geq 0$, define

$$\beta_s = B_{T_x + s} - B_{T_x}, \quad D_s = \inf\{t > s : B_t = 0\}.$$

Note that $t \mapsto L_t$ is constant on $[\, T_x, \, T_x + R_x]$, and that ξ_x (being $\mathcal{F}_{T_x}$-measurable) is independent of $(\beta_s)_{s \geq 0}$. We sometimes write $\beta_s^x := B_{T_x + s} = x + \beta_s$, a Brownian motion started at x. By expanding

$$M_{T_x + (t \wedge R_x)} - M_{T_x}$$

in two different ways using (2.3) and (2.4) we obtain

$$\int_0^{t \wedge R_x} h(x + \beta_u, \xi_x) \, d\beta_u = H(x + \beta_{t \wedge R_x}, \xi_x) - H(x, \xi_x) \quad \forall t \geq 0 \text{ a.s.}. \tag{2.5}$$

The stochastic integral in (2.5) is a local martingale, hence so is the right side. However this is a local martingale function of (absorbed) Brownian motion and thus is an affine function of that Brownian motion. Hence, almost surely for all $t \in [0, R_x]$,

$$H(x+\beta_t, \xi_x) - H(x, \xi_x) = f(x, \xi_x) \cdot \beta_t + g(x, \xi_x) = \int_0^t f(x, \xi_x)\, d\beta_u + g(x, \xi_x), \tag{2.6}$$

for certain functions f and g. Taking $t = 0$ we find that $g(x, \xi_x)$ is identically 0. Thus, almost surely for all $t \in [0, R_x]$,

$$H(x+\beta_t, \xi_x) - H(x, \xi_x) = \int_0^t f(x, \xi_x)\, d\beta_u. \tag{2.7}$$

We now verify that $f(x, y)$ does not depend on x. To this end fix $x \neq 0$ and note that, by the independence of β and ξ_x, and the fact that ξ_x has a density with respect to Lebesgue measure, we obtain

$$\int_0^{t \wedge R_x} h(x+\beta_u, y)\, d\beta_u = \int_0^{t \wedge R_x} f(x, y)\, d\beta_u, \tag{2.8}$$

for a.e. $y \in [0, \infty)$ with respect to Lebesgue measure. Now suppose that $x > 0$. Fix x' such that $0 < x' < x$ and define the following:

$$T^{'} := \inf(u > T_x : B_u = x'), \qquad \beta_s^{'} := B_{T'+s}.$$

Exploiting the fact that the β' path is a fragment of the β path:

$$\beta_s^{'} = \beta_{s+R}^x,$$

where $R := (T^{'} - T_x)$, we conclude that for $0 \leq t \leq (U_x - T^{'})$,

$$\int_0^t f(x', y)\, d\beta_u^{'} = \int_0^t f(x, y)\, d\beta_u^{'}, \tag{2.9}$$

which implies that for $t \in [0, U_x - T']$ there exists a Lebesgue null set N_x such that for all $y \in [0, \infty) \setminus N_x$ we have

$$f(x', y) = f(x, y), \qquad \forall x' \in]0, x[. \tag{2.10}$$

We now let x vary through $\mathbf{N}$ to conclude that there exists a single function $f_+ : [0, \infty) \to \mathbf{R}$, uniquely determined on $[0, \infty) \setminus \bigcup_{n \in \mathbf{N}} N_n$, such that

$$H(x+\beta_t, \xi_x) - H(x, \xi_x) = \int_0^t f_+(\xi_x)\, d\beta_u = f_+(\xi_x) \cdot \beta_t, \tag{2.11}$$

for $0 \le t \le R_x$, a.s. A similar argument holds for $x < 0$ and we thus obtain for $t \in [0, R_x]$,

$$H(x+\beta_t, L_{T_x}) - H(x, L_{T_x}) = \begin{cases} \int_0^t f_+(L_{T_x})\, d\beta_u, & \text{for } x = \beta_0^x > 0, \\ \int_0^t f_-(L_{T_x})\, d\beta_u, & \text{for } x = \beta_0^x < 0, \end{cases} \tag{2.12}$$

where we have denoted by f_+, and f_- the functions corresponding to excursions above and below 0 respectively, and $\xi_x = L_{T_x}$.

The above considerations have involved the *first* excursion of B (away from 0) to reach the level x, but they apply to subsequent excursions as well. A little thought now shows that (2.12) implies (via the Markov property and the fact that the local time is constant over each excursion) the following: For each fixed $r \ge 0$, almost surely on $\{B_r \neq 0\}$,

$$H(B_s, L_s) - H(B_r, L_r) = \begin{cases} f_+(L_s)(B_s - B_r), & \text{for } B_r > 0, \\ f_-(L_s)(B_s - B_r), & \text{for } B_r < 0, \end{cases} \tag{2.13}$$

for all $s \in [r, D_r]$, where D_r is as before: $D_r = \inf\{t > r : B_t = 0\}$. By (2.13) and an approximation argument we may then conclude that

$$H(B_t, L_t) - H(0, L_t) = B_t^+ \cdot f_+(L_t) - B_t^- \cdot f_-(L_t), \quad \forall t \ge 0, \text{ a.s.} \tag{2.14}$$

By (2.4) and the above discussion, we have computed the Itô integrand η;

$$\eta_s = f_+(L_s) \cdot 1_{\{B_s > 0\}} + f_-(L_s) \cdot 1_{\{B_s < 0\}} \text{ for a.e. pair } (s, \omega). \tag{2.15}$$

In view of (1.1) the local time L enjoys the following symmetry property:

$$(B., L.) \stackrel{d}{=} (-B., L.)\,. \tag{2.16}$$

From (2.16), (1.5), and the fact that $(H(B_t, L_t))_{t \ge 0}$ is a continuous local martingale we now deduce that both f_+, and f_- are in L^1_{loc}. This in turn allows us to apply (1.8) to conclude that

$$B_t^+ \cdot f_+(L_t) = \int_0^t f_+(L_s) \cdot 1_{\{B_s > 0\}}\, dB_s + \frac{1}{2} F_+(L_t) \tag{2.17}$$

and

$$B_t^- \cdot f_-(L_t) = -\int_0^t f_-(L_s) \cdot 1_{\{B_s < 0\}}\, dB_s + \frac{1}{2} F_-(L_t), \tag{2.18}$$

where $F_\pm(y) = \int_0^y f_\pm(u)\, du$. Therefore the following holds,

$$\begin{aligned} H(B_t, L_t) - H(0, L_t) &= f_+(L_t) \cdot B_t^+ - f_-(L_t) \cdot B_t^-, \\ &= \int_0^t f_+(L_s) \cdot 1_{\{B_s > 0\}}\, dB_s + \int_0^t f_-(L_s) \cdot 1_{\{B_s < 0\}}\, dB_s \\ &\quad + \frac{1}{2}\left(F_+(L_t) - F_-(L_t)\right) \end{aligned} \tag{2.19}$$

for all $t \geq 0$, almost surely. This implies that

$$N_t := H(0, L_t) + \frac{1}{2}\left[F_+(L_t) - F_-(L_t)\right], \qquad t \geq 0, \tag{2.20}$$

is a continuous local martingale. Since the local time L is flat off the zero set of B, the local martingale N must have vanishing quadratic variation, hence it must be constant in time. Consequently,

$$-H(0, L_t) = \frac{1}{2}\left[F_+(L_t) - F_-(L_t)\right], \qquad \forall t \geq 0, \text{ a.s.}, \tag{2.21}$$

yielding the main assertion in Theorem (1.16). ∎

Proof of Corollary (1.19).

Let $M_t = H(B_t, L_t)$ be a continuous local martingale vanishing on the zero set $\mathcal{Z}$ of B. By Theorem (1.16) we have

$$0 = F_+(L_t) - F_-(L_t), \qquad t \in \mathcal{Z}, \tag{2.22}$$

almost surely. But as t increases through $\mathcal{Z}$, the local time process L increases continuously from 0 to $+\infty$. In other words, $L(\mathcal{Z}) = [0, \infty[$, almost surely. It follows from this and (2.22) that $F_+(x) = F_-(x)$ for all $x \geq 0$. Consequently, $f_+ = f_-$ Lebesgue a.e. on $[0, \infty[$. Define $f := f_+$. Since the distribution of L_t has a density with respect to Lebesgue measure, we have $f(L_t) = f_+(L_t) = f_-(L_t)$ almost surely, for each fixed $t > 0$. Feeding this into (1.17) and (1.18) we deduce that

$$H(B_t, L_t) = B_t \cdot f(L_t), \tag{2.23}$$

first for each fixed $t > 0$, and then almost surely for all $t \geq 0$ since both sides of (2.23) are continuous in t. ∎

References

[AY] J. Azéma and M. Yor: Une solution simple au problème de Skorokhod. In *Séminaire de Probabilités, XIII*, pp. 90–115. Lecture Notes in Math. **721**, Springer, Berlin, 1979.

[OY] J. Obłój and M.Yor: On local martingale and its supremum: harmonic functions and beyond, In *From Stochastic Calculus to Mathematical Finance*, pp. 517–533. Springer, Berlin, 2006.

[O] J. Obłój: A complete characterization of local martingales which are functions of Brownian motion and its maximum, *Bernoulli* **12** (2006) 955–969.

[RVY] B. Roynette, P. Vallois, M. Yor: Limiting laws associated with Brownian motion perturbed by its maximum, minimum and local time. II, *Studia Sci. Math. Hungar.* **43** (2006) 295–360.

[W] D.M. Wroblewski: Non-smooth Brownian martingales and stochastic integral representations, Ph. D. thesis, University of California San Diego, 2007.

[Y] M. Yor: Sur le balayage des semi-martingales continues. In *Séminaire de Probabilités, XIII*, pp. 453–471. Lecture Notes in Math. **721**, Springer, Berlin, 1979.