A New Approach to the Martingale Representation Theorem

by

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Abstract: Let $(W_t)_{t\geq 0}$ be a Brownian motion. We represent the distribution-valued process (δ_{W_t}) as the solution of an evolution equation. Using this we prove the Martingale Representation Theorem, with an explicit expression for the integrand for random variables of the form $f_1(W_{t_1})f_2(W_{t_2}).....f_n(W_{t_n})$. We introduce a new stochastic Sobolev space and reformulate the Martingale Representation theorem in terms of elements from this space.

0. Introduction.

In this paper we develop a new approach to the Martingale Representation Theorem for Brownian motion [**11**]. This approach connects two different streams of the literature on infinite-dimensional Stochastic Analysis: The one deals with representation of solutions of "stochastic evolution equations" (so-called "mild solutions") ([**5**],[**14**] [**6**],[**4**],[**9**],[**8**]), the other concerns the explicit description of the integrand in the stochastic integral representation of functionals of Brownian motion ([**3**],[**17**],[**15**],[**7**],[**1**], [**19**], [**20**]).

While stochastic evolution equations have been studied in fairly general settings in the references cited above, we deal here with the simple but basic example of the Brownian motion. Our starting point is the equation satisfied by the process (δ_{W_t}), where δ_x is the Dirac distribution at $x \in \mathbf{R}^d$ and (W_t) is a d-dimensional Brownian motion. It was shown in [**21**] that this $\mathcal{S}'$-valued process satisfies the following equation in a Hilbert space $\mathcal{S}'_p$ contained in $\mathcal{S}'$:

$$\delta_{W_t} = \delta_{W_0} - \sum_{i=1}^{d} \int_0^t \partial_i(\delta_{W_s})\, dW_s^i + \frac{1}{2}\sum_{i=1}^{d} \int_0^t \partial_i^2(\delta_{W_s})\, ds$$

This is clearly a stochastic evolution equation of the form

$$\begin{aligned} dY_t &= AY_t\, dt + dZ_s \\ Y_0 &= \delta_{W_0} \end{aligned} \tag{0.1}$$

where $A = \frac{1}{2}\Delta : \mathcal{S} \to \mathcal{S}$ is the infinitesimal generator of the Brownian semigroup (T_t), and $Z_t := -\sum_{i=1}^{d}\int_0^t \partial_i(\delta_{W_s})\, dW_s^i$ is an $\mathcal{S}'$-valued semimartingale. We can speak of the mild form of (0.1), namely

$$Y_t = T_t(Y_0) + \int_0^t T_{t-s}\, dZ_t. \tag{0.2}$$

We prove such a representation for $Y_t := \delta_{W_t}$ in Section 2. The necessary estimates on the semigroup and infinitesimal generator are obtained in Section 1.

The above representation as a mild solution is closely connected with the Martingale Representation Theorem. Indeed, allowing the two sides of (0.2) to act on a test function $f \in \mathcal{S}$, we obtain the well-known representation (when $d = 1$) of $f(W_t)$ (see for example [**2**]):

$$f(W_t) = T_t f(W_0) + \int_0^t \partial(T_{t-s} f)(W_s)\, dW_s. \tag{0.3}$$

In Section 3, we prove the Martingale Representation Theorem for Brownian motion by extending (0.3) to random variables of the form $f_1(W_{t_1}) \ldots . f_n(W_{t_n})$. For such random variables we obtain in Lemma (3.1) an explicit description of the integrand (also observed in [**10**]). While the explicit descriptions of the integrand in [**3**] and [**17**] are on path space, our representation is obtained on an arbitrary probability space supporting the driving Brownian motion.

In section 4, we provide a new description of the integrand in the Martingale Representation Theorem, generalising the description given in section 3 to the whole of L^2. We recall that the usual description of the integrand is in terms of stochastic or Malliavin derivatives (the Clark-Ocone formula). In contrast, we introduce (stochastic) Sobolev spaces that can be defined on an arbitrary probability space on which the Brownian motion is defined. The elements here are functions of (t, ω, x) where $\omega \in \Omega$, and $x \in \mathcal{R}^d$, the state space of the Brownian motion. We show that for any $F \in L^2$ there exists a $\beta^F(t, \omega, x)$ in the appropriate Sobolev space , such that the integrand in the Martingale representation theorem is given as $\nabla \beta^F(t, \omega, W_t)$, where the gradient is taken in the variable $x \in \mathcal{R}^d$.

1. Estimates on the Brownian Semigroup.

Let $p_t(x)$ denote the Gaussian kernel defined by

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}, \quad x \in \mathbf{R}^d, t > 0.$$

For $f \in \mathcal{S} = \mathcal{S}(\mathbf{R}^d)$, the space of rapidly decreasing smooth real-valued functions on $\mathbf{R}^d$, define

$$T_t f(x) := \begin{cases} f * p_t(x), & x \in \mathbf{R}^d, t > 0; \\ f(x), & x \in \mathbf{R}^d, t = 0. \end{cases}$$

Let $\Delta f(x) = \sum_{i=1}^d \dfrac{\partial^2 f}{\partial x_i^2}(x)$ be the Laplacian acting on f, and write $\hat{f}(x) = \int e^{-i(t,x)} f(t)\, dt$ for the Fourier transform of f.

Let $\{h_k : k \in \mathbf{N}_0^d\}$ be the orthonormal basis of $L^2 = L^2(\mathbf{R}^d, dx)$ consisting of the Hermite functions. (Here $\mathbf{N}_0 := \{0, 1, 2, \ldots\}$.) Let $(\cdot, \cdot\cdot)_0$ denote the inner product in L^2. (Throughout the paper we consider only real-valued functions.) For $p \in \mathbf{R}$ and $f \in \mathcal{S}(\mathbf{R}^d)$ define

$$\|f\|_p^2 := \sum_{j \in \mathbf{N}_0^d} (2|j| + d)^{2p} (f, h_j)_0^2,$$

where $|j| := j_1 + j_2 + ... + j_d$ for $j = (j_1, ..., j_d) \in \mathbf{N}_0^d$. The Hilbert space $\mathcal{S}_p$ is the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_p$. It is well known that $\mathcal{S} = \cap_{p>0}\mathcal{S}_p$ and $\mathcal{S}' = \cup_{p\geq 0}\mathcal{S}_p'$, where $\mathcal{S}'$ denotes the space of tempered distributions and $\mathcal{S}_p'$ denotes the dual of $\mathcal{S}_p$. It is also well known that $\mathcal{S}_p' = \mathcal{S}_{-p}$; see [**12**].

The usual topology on $\mathcal{S}$ is given by the family of semi-norms $\|\cdot\|_n^*$ defined by

$$\|f\|_n^* = \max_{0\leq|k|\leq n} \sup_{x\in\mathbf{R}^d} (1+|x|^2)^{n/2}|\partial^k f(x)|.$$

The two families $\{\|\cdot\|_n : n \geq 0\}$ and and $\{\|.\|_n^* : n \geq 0\}$ of semi-norms on $\mathcal{S}$ are equivalent, as we note in the following proposition drawn from [**21**]:

Proposition 1.1.

(a) *Given $n \geq 0$, there exist an integer $m > n$ and a constant $C_1(n)$ such that*

$$\|f\|_n^* \leq C_1(n)\|f\|_m, \qquad \forall f \in \mathcal{S}.$$

(b) *Given $n \geq 0$, there exist an integer $m > n$ and a constant $C_2(n)$ such that*

$$\|f\|_n \leq C_2(n)\|f\|_m^*, \qquad \forall f \in \mathcal{S}.$$

In the following proposition, we put together some properties of the semigroup (T_t) acting on the spaces $\mathcal{S}_p$ (see also [**22**]).

Proposition 1.2. *Given $p > 0$ and $T > 0$, there exists $q > p$ such that the following properties hold:*

(a) *For $t \leq T$, the operator T_t extends to a bounded linear operator from $\mathcal{S}_p'$ to $\mathcal{S}_q'$, and*

$$\sup_{t\leq T}\|T_t\|_{\mathcal{L}(\mathcal{S}_p',\mathcal{S}_q')} < \infty$$

where the norm is the operator norm (defining the strong operator topology) in $\mathcal{L}(\mathcal{S}_p', \mathcal{S}_q')$, the space of bounded linear operators from $\mathcal{S}_p'$ to $\mathcal{S}_q'$.

(b) *If $0 \leq t, s, t+s \leq T$ then for $\phi \in \mathcal{S}_p'$,*

$$T_{t+s}\phi = T_t(T_s\phi),$$

the equality holding in $\mathcal{S}_q'$.

(c) *Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\left\|(T_t - I) - \frac{t}{2}\Delta\right\|_{\mathcal{L}(\mathcal{S}_p',\mathcal{S}_q')} \leq t\epsilon$$

for all $0 \leq t \leq \delta$*.*

(d) *The mapping* $t \mapsto T_t$ *from* $[0,T]$ *to* $\mathcal{L}(\mathcal{S}'_p, \mathcal{S}'_q)$ *is continuous in the strong operator topology.*

Proof. In proving (a) and (b) we shall obtain different values for q. But since $\mathcal{S}'_{q_1} \hookrightarrow \mathcal{S}'_{q_2}$ if $q_1 < q_2$, the larger of the two will satisfy the conditions of the theorem. The quantities $p > 0$ and $T > 0$ are fixed throughout the proof.

(a) It is sufficient to show that there exists $q > p$ and a constant $C > 0$ (depending only on T) such that

$$\|T_t f\|_p \leq C\|f\|_q, \qquad \forall t \in [0,T], \tag{1.1}$$

for all $f \in \mathcal{S}_q$. Then,

$$\begin{aligned}\|T_t\|_{L(\mathcal{S}'_p,\mathcal{S}'_q)} &= \sup_{\|f\|_{-p}\leq 1} \|T_t f\|_{-q} = \sup_{\|f\|_{-p}\leq 1} \sup_{\|\phi\|_q \leq 1} |\langle T_t f, \phi\rangle| \\ &\leq \sup_{\|f\|_{-p}\leq 1} \sup_{\|\phi\|_q \leq 1} |\langle f, T_t\phi\rangle| \leq \sup_{\|f\|_{-p}\leq 1} \sup_{\|\phi\|_q \leq 1} \|f\|_{-p}\|T_t\phi\|_p \\ &\leq C.\end{aligned}$$

We remark that the duality $\langle f, \phi\rangle$ between $f \in \mathcal{S} \subset \mathcal{S}_p$ and $\phi \in \mathcal{S}'_p$ is the same as the distribution duality between f and ϕ. By Proposition 1.1 it is sufficient to show that, for a given a non-negative integer m, there exist an integer $n > m$ and a constant $C > 0$ such that

$$\|T_t f\|^*_m \leq C\|f\|^*_n, \qquad \forall t \in [0,T].$$

Since the Fourier transform $\hat{}: \mathcal{S}_c \to \mathcal{S}_c$ ($\mathcal{S}_c$ denoting the complexification of $\mathcal{S}$) is a homeomorphism, we have

$$\begin{aligned}\|T_t f\|^*_m &\leq C_0\|\widehat{(T_t f)}\|^*_{m+m_1} \leq C_0\|\widehat{p_t}\hat{f}\|^*_{m+m_1} \\ &\leq C_1(T)\|\hat{f}\|^*_{m+m_1+m_2} \leq C_2(T)\|f\|^*_n\end{aligned}$$

where $n = m + m_1 + m_2 + m_3$ for some integers $m_i > 0$, $i = 1,2,3$.

(b) The semigroup property $T_{t+s}\phi = T_t T_s\phi$ is well known for $\phi \in \mathcal{S}$. In general, given $p > 0$ and $T > 0$ we choose q such that if $0 \leq t, s, t+s \leq T$, then $T_s\phi$ and $T_t T_s\phi$ belong to $\mathcal{S}'_q$ for $\phi \in \mathcal{S}'_p$ (such a choice is possible by (a)). By continuity , we have $T_{t+s}\phi = T_t T_s\phi$ for all $\phi \in \mathcal{S}'_p$.

(c) As in the proof of (a) it suffices to show that given $T > 0$ and integral $m \geq 0$, there exists an integer $n > m$ such that the following holds: for all $\epsilon > 0$ there exists $\delta > 0$ with

$$\|\hat{\psi}_t\|^*_m \leq t\epsilon\|\hat{\phi}\|^*_n$$

for all $0 \leq t < \delta$ and for all $\phi \in \mathcal{S}$, where $\psi_t := (T_t - I)\phi - \frac{t}{2}\Delta\phi$. Notice that

$$\hat{\psi}_t(x) = \left(\hat{p}_t(x) - 1 + t\frac{1}{2}|x|^2\right)\hat{\phi}(x)$$

where $\hat{p}_t(x) = e^{-t|x|^2/2}$. Define $\eta(u) = e^{-u/2} - 1 + \frac{u}{2}$, $u \in \mathbf{R}$, and $\eta_t(x) = \eta(t|x|^2)$, $x \in \mathbf{R}^d$. We then have

$$\hat{\psi}_t(x) = \eta_t(x)\hat{\phi}(x).$$

Hence ,

$$\|\hat{\psi}_t\|_m^* = \max_{|k|\le m} \sup_{x\in\mathbf{R}^d} \left[(1+|x|^2)^{m/2}|\partial^k(\eta_t\hat{\phi})(x)|\right].$$

Since

$$\partial^k(\eta_t\hat{\phi})(x) = \sum_{|\alpha|\le|k|} C_\alpha(\partial^{k-\alpha}\eta_t)(x)(\partial^\alpha\hat{\phi})(x),$$

it suffices to show that there exists $\delta > 0$ such that, for $|k| \le m$,

$$|\partial^k\eta_t(x)| \le t\epsilon Q_k(x), \tag{1.2}$$

where Q_k is a polynomial. From the definition of η_t we have

$$\partial^k\eta_t(x) = \sum_{\ell=0}^{|k|}\left(\frac{t}{2}\right)^\ell P_\ell(x)\frac{d^\ell\eta}{du^\ell}(t|x|^2),$$

where P_ℓ is a polynomial. Because $\sup_{\ell\ge2,u\ge0}\left|\frac{d^\ell\eta(u)}{du^\ell}\right| < \infty$, it suffices to show that, given $\epsilon > 0$, there exists $\delta > 0$ and $0 < C < \infty$ such that for $t \in (0,\delta)$,

$$\left|\eta'(t|x|^2)\right| \le \epsilon\frac{|x|^2}{4} \tag{1.3}$$

and

$$|\eta(t|x|^2)| \le t\epsilon\cdot C|x|^4. \tag{1.4}$$

The estimate (1.3) follows by an easy calculation while (1.4) is a consequence of the simple inequality $0 \le \eta(u) \le u^2/8$, $u \ge 0$.

(d) Given $p > 0$ and $T > 0$ choose $q_1 > p$ such that

$$\|T_t\|_{\mathcal{L}(\mathcal{S}_{q_1},\mathcal{S}_p)} \le C$$

for all $0 \le t \le T$. That q_1 can be so chosen follows from equation (1.1). Also, the proof of (c) shows that the following (weaker) statement holds : Given $q_1 > 0$, there exists $q > q_1$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|(T_t - I)\|_{\mathcal{L}(\mathcal{S}_q,\mathcal{S}_{q_1})} \le \epsilon$$

for all $0 \le t \le \delta$. With q as above, $0 \le s \le t \le T$, and $|t-s| \le \delta$, we have

$$\begin{aligned}
\|T_t - T_s\|_{\mathcal{L}(\mathcal{S}'_p, \mathcal{S}'_q)} &= \sup_{f \in \mathcal{S}, \|f\|_{-p} \le 1} \|T_t f - T_s f\|_{-q} \\
&= \sup_{f \in \mathcal{S}, \|f\|_{-p} \le 1} \sup_{\|\phi\|_q \le 1} |\langle T_t f - T_s f, \phi\rangle| \\
&= \sup_{f \in \mathcal{S}, \|f\|_{-p} \le 1} \sup_{\|\phi\|_q \le 1} |\langle f, (T_t - T_s)\phi\rangle| \\
&= \sup_{f \in \mathcal{S}, \|f\|_{-p} \le 1} \sup_{\|\phi\|_q \le 1} \|f\|_{-p} \|(T_t - T_s)\phi\|_p \\
&\le \sup_{\|\phi\|_q \le 1} \|T_s(T_{t-s} - I)\phi\|_p \le \epsilon C
\end{aligned}$$

$\square$

2. Mild Solutions.

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which is defined an $\mathbf{R}^d$-valued Brownian motion $(W_t)_{t \ge 0}$. For $0 \le t < \infty$ define the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_\infty$ as follows:

$$\mathcal{F}_t = \sigma\{W_s; 0 \le s \le t\}, \qquad \mathcal{F}_\infty := \sigma\{W_s; 0 \le s < \infty\}.$$

Let δ_x denote the Dirac distribution at $x \in \mathbf{R}^d$. There exists $p > 0$ such that the $\mathcal{S}'$-valued processes (δ_{W_t}), $(\Delta\delta_{W_t})$, and $(\partial_i \delta_{W_t})$ are in fact $\mathcal{S}'_p$-valued continuous $(\mathcal{F}_t)$-adapted processes, and the following Itô Formula holds in $\mathcal{S}'_p$: almost surely, for all $t \ge 0$,

$$\delta_{W_t} = \delta_0 - \sum_{i=1}^d \int_0^t \partial_i(\delta_{W_s})\, dW_s^i + \frac{1}{2} \sum_{i=1}^d \int_0^t \partial_i^2(\delta_{W_s})\, ds. \tag{2.1}$$

See [**21**]. Let $Z_t := -\sum_{i=1}^d \int_0^t \partial i \delta_{W_s}\, dW_s^i$. We refer to [**16**] for definitions and other properties of Hilbert space valued stochastic integrals. Note that (Z_t) is an $\mathcal{S}'_p$-valued continuous $(\mathcal{F}_t)$ local martingale. Fix $p > 0$ as above, and $T > 0$. The following theorem presents δ_{W_t} as the mild solution of the "evolution equation"

$$dY_t = \frac{1}{2}\Delta(Y_t)dt - \nabla(Y_t) \cdot dW_t \tag{2.2}$$

Theorem 2.1. *Fix $p > 0$ such that (2.1) holds in $\mathcal{S}'_p$. Then there exists $q > p$ such that the following equation holds in $\mathcal{S}'_q$: for each $t \ge 0$, almost surely,*

$$\delta_{W_t} = T_t(\delta_{W_0}) - \sum_{i=1}^d \int_0^t T_{t-s}(\partial_i \delta_{W_s})\, dW_s^i = T_t(\delta_{W_0}) + \int_0^t T_{t-s}\, dZ_s \tag{2.3}$$

To prove the theorem, we need the following lemma, in which $Y_t := \delta_{W_t}$.

Lemma 2.1. *There exists $q > p$ such that the following equality holds in $\mathcal{S}_q'$ for each $t \in [0, T]$: almost surely, for $0 \le u \le t$,*

$$\int_0^u T_{t-s}\Delta(Y_s)\,ds = -\left(T_{t-u}(Y_u) - T_t(Y_0) - \int_0^u T_{t-s}\,dY_s\right).$$

Proof. From the properties of the semigroup (T_t) proved in Proposition 1.1 and the properties of the Hilbert space valued stochastic integral, all of the processes involved in the statement of the lemma take values in $\mathcal{S}_q'$. For each n, choose a partition $0 = s_0^n < s_0^n < \cdots < s_m^n = u \le t \le T$, such that $\lim_{n\to\infty} \sup_k |s_k^n - s_{k-1}^n| = 0$. Then the Riemann sum

$$\sum_{k=1}^m T_{t-s_{k-1}^n}(Y_{s_k^n} - Y_{s_{k-1}^n})$$

converges in probability to $\int_0^u T_{t-s}\,dY_s$, in $\mathcal{S}_q'$. On the other hand, performing a summation-by-parts we find that

$$\begin{aligned}
\sum_{k=1}^m T_{t-s_{k-1}^n}(Y_{s_k^n} - Y_{s_{k-1}^n}) &= T_{t-s_{m-1}^n}(Y_u) - T_t(Y_0) - \sum_{k=1}^{m-1}(T_{t-s_k^n} - T_{t-s_{k-1}^n})(Y_{s_k^n}) \\
&= T_{t-s_{m-1}^n}(Y_u) - T_t(Y_0) + \sum_{k=1}^{m-1} T_{t-s_k^n}(T_{s_k^n - s_{k-1}^n} - I)(Y_{s_k^n}) \\
&\quad - \sum_{k=1}^{m-1} T_{t-s_k^n}(\Delta(Y_{s_k^n}))(s_k^n - s_{k-1}^n) + \sum_{k=1}^{m-1} T_{t-s_k^n}(\Delta(Y_{s_k^n}))(s_k^n - s_{k-1}^n) \\
&= T_{t-s_{m-1}^n}(Y_u) - T_t(Y_0) + \sum_{k=1}^{m-1} T_{t-s_k^n}(\Delta(Y_{s_k^n}))(s_k^n - s_{k-1}^n) \\
&\quad + \sum_{k=1}^{m-1} T_{t-s_k^n}\big(T_{s_k^n - s_{k-1}^n} - I - (s_k^n - s_{k-1}^n)\Delta\big)(Y_{s_k^n})
\end{aligned}$$

The last sum converges to zero pointwise in $\mathcal{S}_q'$. Indeed, by Proposition 1.1 (a) and (c), given $\epsilon > 0$ there exists $\delta > 0$ such that if $\sup_k |s_k^n - s_{k-1}^n| < \delta$ then

$$\begin{aligned}
&\Big\| \sum_{k=1}^{m-1} T_{t-s_k^n}\left(T_{s_k^n - s_{k-1}^n} - I - (s_k^n - s_{k-1}^n)\Delta(Y_{s_k^n})\right) \Big\|_{-q} \\
&\qquad \le C \sum_{k=1}^{m-1} \|T_{t-s_k^n}\big(T_{s_k^n - s_{k-1}^n} - I - s_k^n - s_{k-1}^n)\Delta\big)(Y_{s_k^n})\|_{-q} \\
&\qquad \le \epsilon\, C \sup_{s \le t} \|Y_s\|_{-p}.
\end{aligned}$$

The lemma follows by letting $n \to \infty$. ∎

Proof of Theorem 2.1. We start with a Riemann-sum approximation of the stochastic integral $\int_0^t T_{t-s} dZ_s$. Choose a sequence of partitions $0 = s_0^n < \cdots < s_m^n = t$ of $[0,t]$ with $\lim_{n\to\infty} \sup_{1\le k\le m(n)} |s_k^n - s_{k-1}^n| = 0$. Let $Y_t = \delta_{W_t}$. From equation (2.1) we have

$$\begin{aligned}
\sum_{k=1}^m T_{t-s_{k-1}^n}\Big(Z_{s_k^n} - Z_{s_{k-1}^n}\Big) &= \sum_{k=1}^m T_{t-s_{k-1}^n}\Big(Y_{s_k^n} - Y_{s_{k-1}^n} - \int_{s_{k-1}^n}^{s_k^n} \Delta(Y_s)\, ds\Big) \\
&= \sum_{k=1}^m \Big[T_{t-s_{k-1}^n}(Y_{s_k^n} - Y_{s_{k-1}^n}) - \int_{s_{k-1}^n}^{s_k^n} T_{t-s_{k-1}^n}\Delta(Y_s)\, ds\Big] \\
&= \sum_{k=1}^m \Big(T_{t-s_{k-1}^n}(Y_{s_k^n} - Y_{s_{k-1}^n})\Big) + \sum_{k=1}^m \int_{s_{k-1}^n}^{s_k^n} (T_{t-s} - T_{t-s_{k-1}^n})\Delta(Y_s)\, ds \\
&\quad - \sum_{k=1}^m \int_{s_{k-1}^n}^{s_k^n} T_{t-s}\Delta(Y_s)\, ds \\
&= \sum_{k=1}^m \Big(T_{t-s_{k-1}^n}(Y_{s_k^n} - Y_{s_{k-1}^n})\Big) + \sum_{k=1}^m \int_{s_{k-1}^n}^{s_k^n} (T_{t-s} - T_{t-s_{k-1}^n})\Delta(Y_s)\, ds \\
&\quad + \sum_{k=1}^m \Big(T_{t-s_k^n} Y_{s_k^n} - T_{t-s_{k-1}^n} Y_{s_{k-1}^n} - \int_{s_{k-1}^n}^{s_k^n} T_{t-s} dY_s\Big),
\end{aligned}$$

where the last equality follows from Lemma 2.1. We now define, for each n, the operator-valued simple function T_s^n as follows :

$$T_s^n := \begin{cases} T_{t-s_{k-1}^n}, & s_{k-1}^n \le s < s_k^n,\ k = 1,\dots,m; \\ T_{t-s_{m-1}^n}, & s = s_m^n = t. \end{cases}$$

In view of the above calculations

$$\sum_{k=1}^m T_{t-s_{k-1}^n}\Big(Z_{s_k^n} - Z_{s_{k-1}^n}\Big) = \int_0^t (T_s^n - T_{t-s})\, dY_s + \int_0^t (T_{t-s} - T_s^n)\Delta Y_s\, ds + Y_t - Y_0. \tag{2.4}$$

By the continuity of $t \to T_t : [0,T] \to \mathcal{L}(\mathcal{S}_p', \mathcal{S}_q')$,

$$\sup_{s\le t} \|T_s^n - T_{t-s}\|_{\mathcal{L}(\mathcal{S}_p', \mathcal{S}_q')} \to 0$$

as $n \to \infty$. Hence the two integrals on the right side of (2.4) converge to zero in $\mathcal{S}_q'$, in probability. The left side converges to $\int_0^t T_{t-s}\, dZ_s$ in $\mathcal{S}_q'$, in probability. The proof of Theorem 2.1 is complete.

∎

3. The Martingale Representation Theorem.

In this section we prove the Martingale Representation Theorem for square-integrable Wiener functionals. In Lemma 3.1 below, we use Theorem 2.1 to obtain an explicit representation of the integrand for a class of (finite dimensional) functionals defined as follows:

$$\mathcal{C} = \Big\{f_1(W_{t_1})\cdots f_m(W_{t_m}) : m \ge 1, f_i \in \mathcal{S}, 0 \le t_1 < t_2 < \cdots < t_m < \infty\Big\}$$

For such a functional $F = F(f_1, \dots, f_m; t_1, \dots, t_m)$ define the $\mathcal{S}$-valued simple process $(\alpha_s^F)_{s\geq 0}$ as follows:

$$
\begin{aligned}
\alpha_s^F(x) := \sum_{k=1}^{m} & f_1(W_{t_1}) \cdots f_{k-1}(W_{t_{k-1}})\, 1_{(t_{k-1},t_k]}(s) \\
& \cdot \Big(T_{t_k - s}(f_k(T_{t_{k+1}-t_k}(f_{k+1}(\cdots(T_{t_m - t_{m-1}}(f_m)\cdots))))\Big)(x).
\end{aligned} \tag{3.1}
$$

Lemma 3.1. *Let $F = \prod_{k=1}^m f_k(W_{t_k})$ be an element of $\mathcal{C}$, as described above. Then*

$$
F = \mathbf{E}[F|\mathcal{F}_0] - \sum_{j=1}^{d} \int_0^\infty \langle \partial_j \delta_{W_s}, \alpha_s^F \rangle \, dW_s^j. \tag{3.2}
$$

Proof. For $m = 1$ we get from Theorem 2.1,

$$
\begin{aligned}
f_1(W_{t_1}) &= \langle \delta_{W_{t_1}}, f_1 \rangle \\
&= \langle T_{t_1} \delta_{W_0}, f_1 \rangle - \sum_{j=1}^{d} \int_0^{t_1} \big\langle T_{t_1 - s} \partial_j \delta_{W_s}, f_1 \big\rangle \, dW_s^j \\
&= \langle \delta_{W_0}, T_{t_1} f_1 \rangle - \sum_{j=1}^{d} \int_0^{\infty} \big\langle \partial_j \delta_{W_s}, 1_{(0,t_1]}(s) T_{t_1 - s} f_1 \big\rangle \, dW_s^j,
\end{aligned}
$$

which is in accord with (3.2). Proceeding by induction, assume that (3.2) is true for a product with no more than $m-1$ factors. Define

$$
Y_1(t) := - \sum_{j=1}^{d} \int_0^{t_{m-1} \wedge t} \big\langle \partial_j \delta_{W_s}, \alpha_s^G \big\rangle \, dW_s^j,
$$

where $G := \prod_{k=1}^{m-1} f_k(W_{t_k})$, and

$$
Y_2(t) := - \sum_{j=1}^{d} \int_{t_{m-1} \wedge t}^{t_m \wedge t} \big\langle \partial_j \delta_{W_s}, T_{t_m - s} f_m \big\rangle \, dW_s^j.
$$

By the induction hypothesis, if $s \geq t_{m-1}$ then

$$
Y_1(s) = Y_1(t_{m-1}) = G - \mathbf{E}[G|\mathcal{F}_0].
$$

Now we apply the case $m = 1$ to the Brownian motion $W_t^{t_{m-1}} := W_{t + t_{m-1}}$, $t \geq 0$. It follows that for $s \geq t_m$

$$
Y_2(s) = Y_2(t_m) = f_m(W_{t_m}) - T_{t_m - t_{m-1}} f_m(W_{t_{m-1}}).
$$

Clearly $Y_2(s) = 0$ for $s \leq t_{m-1}$. Integrating by parts, we obtain

$$
\begin{aligned}
Y_1(t_m) Y_2(t_m) &= - \sum_{j=1}^{d} \int_{t_{m-1}}^{t_m} Y_1(s) \big\langle \partial_j \delta_{W_s}, T_{t_m - s} f_m \big\rangle \, dW_s^j \\
&= - \sum_{j=1}^{d} \int_0^\infty \{G - \mathbf{E}[G|\mathcal{F}_0]\} \big\langle \partial_j \delta_{W_s}, 1_{(t_{m-1}, t_m]}(s) T_{t_m - s} f_m \big\rangle \, dW_s^j
\end{aligned}
$$

On the other hand, by the remarks above ,

$$\begin{aligned}Y_1(t_m)Y_2(t_m) &= \{G - \mathbf{E}[G|\mathcal{F}_0]\} \cdot \left\{f_m(W_{t_m}) - T_{t_m - t_{m-1}} f_m(W_{t_{m-1}})\right\} \\ &= F + \mathbf{E}[G|\mathcal{F}_0] \cdot \left\{T_{t_m - t_{m-1}} f_m(W_{t_{m-1}})\right\} \\ &\qquad - \mathbf{E}[G|\mathcal{F}_0] f_m(W_{t_m}) - G \cdot T_{t_m - t_{m-1}} f_m(W_{t_{m-1}}).\end{aligned} \tag{3.3}$$

We shall examine the last three terms in (3.3) separately. Applying the representation obtained for $m = 1$ to $T_{t_m - t_{m-1}} f_m(W_{t_{m-1}})$ we find that

$$\begin{aligned}&T_{t_m - t_{m-1}} f_m(W_{t_{m-1}}) \\ &\qquad = T_{t_{m-1}} \left(T_{t_m - t_{m-1}} f_m\right)(W_0) - \sum_{j=1}^{d} \int_0^{t_{m-1}} \langle \partial_j \delta_{W_s}, T_{t_{m-1} - s} \left(T_{t_m - t_{m-1}} f_m\right) \rangle \, dW_s^j.\end{aligned}$$

Similarly,

$$\mathbf{E}[G|\mathcal{F}_0] f_m(W_{t_m}) = \mathbf{E}[G|\mathcal{F}_0] \cdot \left\{ T_{t_m} f_m(W_0) - \sum_{j=1}^{d} \int_0^{t_m} \langle \partial_j \delta_{W_s}, T_{t_m - s}(f_m) \rangle \, dW_s^j \right\}.$$

Define $f^*_{m-1}(x) := f_{m-1}(x) T_{t_m - t_{m-1}} f_m(x)$ and $G^* := \left[\prod_{k=1}^{m-2} f_k(W_{t_k})\right] \cdot f^*_{m-1}(W_{t_{m-1}})$. By the induction hypothesis,

$$\begin{aligned}G \cdot T_{t_m - t_{m-1}} f_m(W_{t_{m-1}}) &= G^* f^*_{m-1}(W_{t_{m-1}}) \\ &= E\left[G^* f^*_{m-1}(W_{t_{m-1}}) | \mathcal{F}_0\right] \\ &\qquad - \sum_{j=1}^{d} \int_0^{\infty} \langle \partial_j \delta_{W_s}, \alpha_s^{G^*} \rangle \, dW_s^j \\ &= E\left[f_1(W_{t_1}) \cdots f_m(W_{t_m}) | \mathcal{F}_0\right] \\ &\qquad - \sum_{j=1}^{d} \int_0^{\infty} \langle \partial_j \delta_{W_s}, \alpha_s^{G^*} \rangle \, dW_s^j\end{aligned}$$

where we have used the Markov property of Brownian motion for the last equality. Equating the two expressions for $Y_1(t_m)Y_2(t_m)$, the lemma follows. □

Theorem 3.1. *If $F \in L^2(\mathcal{F}_\infty)$ then there exist previsible processes $(g_j)_{1 \le j \le d}$ satisfying*

$$\sum_{j=1}^{d} \mathbf{E} \int_0^{\infty} g_j^2(s) \, ds \;<\; \infty$$

such that

$$F = \mathbf{E}[F|\mathcal{F}_0] + \sum_{j=1}^{d} \int_0^{\infty} g_j(s) \, dW_s^j$$

Proof. Since $L^2(\mathcal{F}_\infty)$ is the closure of the linear span of $\mathcal{C}$, the representation for $F \in L^2(\mathcal{F}_\infty)$ follows from Lemma 3.1 and the linearity isometry property of the stochastic integral. ∎

4. The Map $F \mapsto \alpha_t^F(\omega, x)$.

The main purpose of this section is to extend the domain of the mapping $F \mapsto \alpha^F$ from $\mathcal{C}$ to all of $L^2(\mathcal{F}_\infty)$. Throughout this section our $\mathbf{R}^d$-valued Brownian motion $(W_t)_{t\geq 0}$ with $W_0 = 0$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $(\mathcal{F}_t)_{t\geq 0}$ is the filtration induced by (W_t).

Proposition 4.1. *Given $F \in \mathcal{C}$, the process $t \mapsto \alpha_t^F(\omega, W_t(\omega))$ is a continuous version of the conditional expectation process $\mathbf{E}[F|\mathcal{F}_t](\omega)$, $t \geq 0$, for $\mathbf{P}$-a.e. $\omega \in \Omega$. In other words, $t \mapsto \alpha_t^F(W_t)$ is the $(\mathcal{F}_t)$ optional projection of the (constant in time) process $t \mapsto F$.*

Proof. The conditional expectation identity

$$\mathbf{E}[F|\mathcal{F}_t](\omega) = \alpha_t^F(\omega, W_t(\omega)) \tag{4.1}$$

(for $\mathbf{P}$-a.e. $\omega \in \Omega$) follows immediately from (3.1) and the simple Markov property of Brownian motion. The continuity of $t \mapsto \alpha_t^F(W_t)$ is an evident consequence of the smoothness of the f_j in (3.1). ∎

The Martingale Representation Theorem can now be reformulated in terms of α^F . To this end we introduce two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$ of processes. Let $\mathcal{P}$ be the previsible σ-field on $(0,\infty)\times\Omega$ and let $\mathcal{B}^d$ be the Borel σ-field on $\mathbf{R}^d$. We say that two $\mathcal{P}\otimes\mathcal{B}^d$-measurable maps $f, g : (0,\infty)\times\Omega\times\mathbf{R}^d \to \mathbf{R}$ are *equivalent* provided $f(t,\omega,W_t(\omega)) = g(t,\omega,W_t(\omega))$ for Leb$\otimes P$-a.e. $(t,\omega) \in (0,\infty)\times\Omega$. Let $\mathcal{H}$ be the collection of equivalence classes of such maps such that

$$\|f\|_{\mathcal{H}}^2 := \mathbf{E}\int_0^\infty |f(s,\cdot,W_s)|^2\,ds < \infty \tag{4.2}$$

for one (or every) member of the equivalence class. Clearly $\mathcal{H}$ is a (real) Hilbert space with respect to the inner product

$$\langle f, g\rangle_{\mathcal{H}} := \mathbf{E}\int_0^\infty f(s,\cdot,W_s)g(s,\cdot,W_s)\,ds.$$

Let $\mathcal{S}_0 = \mathcal{P}_b \otimes \mathcal{S}$ where $\mathcal{P}_b$ is the class of bounded real-valued previsible processes; that is, $\mathcal{S}_0$ is the tensor product of the vector spaces $\mathcal{P}_b$ and $\mathcal{S}$ and consists of finite linear combinations of functions of the form $(f\otimes\varphi)(s,\omega,x) := f(s,\omega)\cdot\varphi(x)$ where $f \in \mathcal{P}_b$ and $\varphi \in \mathcal{S}$.

Proposition 4.2. *$\mathcal{S}_0$ is dense in $\mathcal{H}$.*

Proof. Choose $\varphi_n \in \mathcal{S}$ with $0 \leq \varphi_n \leq 1$, such that

$$\varphi_n(x) = \begin{cases} 1, & |x| \leq n, \\ 0, & |x| \geq n+1. \end{cases}$$

Given $f \in \mathcal{H}$ define $g_n(s,\omega) := I_{(-n,n)}(f(s,\omega,W_s(\omega))) \cdot f(s,\omega,W_s(\omega))$, and $f_n := g_n \otimes \varphi_n$. Then

$$\|f - f_n\|_{\mathcal{H}}^2 = E \int_0^\infty |f(s,\cdot,W_s)|^2 \left[1 - I_{(-n,n)}(f(s,\cdot,W_s))\varphi_n(W_s)\right] ds \\ \to 0, \qquad \text{as } n \to \infty.$$

∎

Now define $\partial_i : \mathcal{S}_0 \to \mathcal{S}_0$, $i = 1, 2, \ldots, d$, as follows: First put $\partial_i(f \otimes \varphi) := f \otimes \partial_i \varphi$, and then extend ∂_i to all of $\mathcal{S}_0$ by linearity. For $\Phi \in \mathcal{S}_0$ define

$$\|\Phi\|_{\mathcal{H}_1}^2 := \|\Phi\|_{\mathcal{H}}^2 + \sum_{i=1}^d \|\partial_i \Phi\|_{\mathcal{H}}^2. \tag{4.3}$$

Let $\mathcal{H}_1$ denote the Hilbert space obtained by completing $\mathcal{S}_0$ with respect to the inner product associated with $\|\cdot\|_{\mathcal{H}_1}$. Notice that $\mathcal{H}_1 \subset \mathcal{H}$.

Proposition 4.3. *The operator ∂_i extends as a bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}$.*

Proof. For $f \in \mathcal{S}_0, \|\partial_i f\|_{\mathcal{H}} \le \|f\|_{\mathcal{H}_1}$. ∎

Let $\nabla f := (\partial_1 f, \ldots, \partial_d f)$ denote the corresponding "gradient" operator.

Theorem 4.1. *There is a linear map $F \mapsto \beta^F$ from $L^2(\mathcal{F}_\infty)$ into $\mathcal{H}_1$ such that*

$$F = \mathbf{E}[F] + \int_0^\infty (\nabla \beta^F)_s(W_s) \cdot dW_s. \tag{4.4}$$

The process β^F is uniquely determined by F.

Proof. Clearly it suffices to prove the representation for $F \in L^2(\mathcal{F}_T)$ for $0 < T < \infty$. We thus restrict our attention to such functionals. Let $\mathcal{V}$ denote the linear span of $\mathcal{C}$, both $\mathcal{V}$ and $\mathcal{C}$ being viewed as subspaces of $L^2(\mathcal{F}_T)$. Evidently $\mathcal{V}$ is dense in $L^2(\mathcal{F}_T)$. The (linear) map $F \mapsto \alpha^F$ extends (uniquely) by linearity to a map $\alpha : \mathcal{V} \to \mathcal{H}_1$. Let us verify that α is continuous. First, by Proposition 4.1, if $F \in L^2(\mathcal{F}_T)$ then

$$\begin{aligned} \|\alpha^F\|_{\mathcal{H}} &= \mathbf{E} \int_0^T |\alpha_t^F(W_t)|^2 \, dt \\ &= \mathbf{E} \int_0^T |\mathbf{E}[F|\mathcal{F}_t]|^2 \, dt \\ &= \int_0^T \|\mathbf{E}[F|\mathcal{F}_t]\|_{L^2(\mathcal{F}_T)}^2 \, dt \\ &\le \int_0^T \|F\|_{L^2(\mathcal{F}_T)}^2 \, dt \\ &= T\|F\|_{L^2(\mathcal{F}_T)}. \end{aligned}$$

Next, from (3.2), a continuous version of the martingale $M_t := \mathbf{E}[F|\mathcal{F}_t] - E[F]$ is given by

$$\int_0^t (\nabla \alpha^F)_s(W_s) \cdot dW_s,$$

and therefore

$$\langle M \rangle_T = \int_0^T |(\nabla \alpha^F)_s(W_s)|^2 \, ds.$$

Consequently,

$$\begin{aligned} &\mathbf{E} \int_0^T |(\nabla \alpha^F)_s(W_s)|^2 \, ds \\ &\qquad = \mathbf{E}[\langle M \rangle_T] \\ &\qquad = \mathbf{E}[M_T^2] = \mathbf{E}[F^2] - \mathbf{E}[F]^2 \le \|F\|^2_{L^2(\mathcal{F}_T)}. \end{aligned}$$

It follows that $\|\alpha^F\|^2_{\mathcal{H}_1} \le (1+T)\|F\|^2_{L^2(\mathcal{F}_T)}$, which implies the asserted continuity.

Let $\beta : F \mapsto \beta^F$ denote the (unique) extension by continuity of α to all of $L^2(\mathcal{F}_T)$. Given $F \in L^2(\mathcal{F}_T)$, let $(F^n) \subset \mathcal{V}$ such that $F^n \to F$ in $L^2(\mathcal{F}_T)$. By the continuity of α just established, $\alpha^{F_n} \to \beta^F$ in $\mathcal{H}_1$. The isometry property of the stochastic integral now ensures that (4.4) holds. The uniqueness assertion is left to the reader.

□

References

[1] Ahn, H.: Semimartingale integral representation. *Ann. Probab.*, **25** (1997) 997–1010.

[2] Biane, Ph.: Chaotic representation for finite Markov chains, *Stochastics Stochastics Rep.* **30** (1990) 61–68.

[3] Clark, J.M.C.: The representation of functionals of Brownian motion by stochastic integrals. *Ann. Math. Stat.* **42** (1971)1282–1295.

[4] Da Prato, G. and Zabczyk, J.: *Stochastic Equations in Infinite Dimensions.*, Encyclopedia of Mathematics and its Applications **44**, Cambridge University Press, Cambridge, 1992.

[5] Dawson, D.A.: Stochastic evolution equations and related measure processes. *J. Mult. Anal.* **5** (1975) 1–52.

[6] Dawson, D.A. and Gorostiza, L.G.: Generalized solutions of stochastic evolution equations. In *Stochastic Partial Differential Equations and Applications, II*, Lecture Notes in Math. **1390**, Springer, Berlin, 1989.

[7] de Faria, M., Oliveira, M.J., and Streit, L.: A generalized Clark-Ocone formula, *Random Oper. Stochastic Equations* **8** (2000) 163–174.

[8] Gawarecki, L., Mandrekar, V., and Richard, P.: Existence of weak solutions for stochastic differential equations and martingale solutions for stochastic semilinear equations. *Random Oper. Stochastic Equations* **3** (1999) 215–240.

[9] Ichikawa, A.: Stability of semi-linear stochastic evolution equations. *J. Math. Anal. Appl.* **90** (1982) 12–44.

[11] Itô, K.: Multiple Wiener integral. *J. Math. Soc. Japan* **3** (1951) 157–169.

[12] Itô, K.: *Foundations of Stochastic Differential Equations in Infinite-dimensional Spaces.*, CBMS-NSF Regional Conference Series in Applied Mathematics, **47**. SIAM, Philadelphia, 1984.

[13] Itô, K: On Malliavin calculus. In *Probability Theory* (Singapore,1989), pp. 47–72. De Gruyter, Berlin, 1992.

[10] Jacod, J., Méléard, S., and Protter, P.: Explicit form and robustness of martingale representations. *The Ann. Probab.* **28** (2000) 1747–1780.

[14] Kallianpur, G. and Pérez-Abreu, V.: Stochastic evolution equations driven by nuclear-space-valued martingales. *Appl. Math. Optim.* **17** (1988) 237–272.

[15] Karatzas, I., Ocone, D.L., and Li, J.: An extension of Clark's formula, *Stochastics Stochastics Rep.* **37** (1991) 127–131.

[16] Métivier, M.: *Semimartingales. A Course on Stochastic Processes*. De Gruyter, Berlin-New York, 1982.

[17] Ocone, D.: Malliavin's calculus and stochastic integral representations of functionals of diffusion processes. *Stochastics* **12** (1984) 161–185.

[18] Pazy, A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springe-Verlag, New York, 1983.

[19] Peccati, G.: A representation result for time-space Brownian chaos, *Ann. Inst. H. Poincaré Probab. Statist.*, **37** (2001) 607–625.

[20] Peccati, G.: Explicit formulae for time-space Brownian chaos, *Bernoulli* **9** (2003) 25–48.

[21] Rajeev, B.: From Tanaka formula to Itô formula: distributions, tensor products and local times, *Séminaire de Probabilités XXXV*, Lecture Notes in Math. **1755** (Springer-Verlag) (2001).

[22] Rajeev, B. and Thangavelu, S.: Probabilistic representations of solutions to the heat equation. *Proc. Indian Acad. Sci. Math. Sci.* **113** (2003) 321–332.

[23] Rozovskii, B.L.: *Stochastic Evolution Systems. Linear Theory and Applications to Nonlinear Filtering.* Kluwer, Dordrecht, 1990.

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