NON-SYMMETRIC PERTURBATIONS OF SYMMETRIC DIRICHLET FORMS

PATRICK J. FITZSIMMONS AND KAZUHIRO KUWAE

ABSTRACT. We provide a path-space integral representation of the semigroup associated with the quadratic form obtained by lower order perturbation of a symmetric local Dirichlet form. The representation is a combination of Feynman-Kac and Girsanov formulas, and extends previously known results in the framework of symmetric diffusion processes through the use of the Hardy class of smooth measures, which contains the Kato class of smooth measures.

1. INTRODUCTION

Let $\mathbf{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbf{P}^x)$ be a symmetric Markov diffusion processes with state space E and symmetry measure m. Let $(\mathcal{E}, \mathcal{D})$ be the associated (strongly local, quasi-regular) Dirichlet form. We examine a class of lower order perturbations of $(\mathcal{E}, \mathcal{D})$, obtaining path-space integral representations of the associated semigroups. These representations are a combination of Feynman-Kac and Girsanov formulas.

Let M and $\hat{M}$ be local martingale continuous additive functionals (CAFs) of $\mathbf{X}$ (denoted $M, \hat{M} \in \mathcal{M}_{\rm loc}$), and let C be a CAF of $\mathbf{X}$ with paths locally of finite variation. The ordered triple $(M, \hat{M}, C)$ is a convenient way of parametrizing the perturbations we have in mind.

The quadratic variation processes $\langle M\rangle$ and $\langle \hat{M}\rangle$ are PCAFs of $\mathbf{X}$, with corresponding smooth Revuz measures $\mu_{\langle M\rangle}$ and $\mu_{\langle \hat{M}\rangle}$, their so-called energy measures. Similarly, C admits a smooth signed measure ν as Revuz measure.

We say that a smooth measure μ is of the *Hardy class* (and write $\mu \in S_H$) provided there are constants $\delta = \delta(\mu) \in]0, \infty[$ and $\gamma = \gamma(\mu) \in [0, \infty[$ such that

$$\int_E \tilde{u}^2\, d\mu \le \delta \cdot \mathcal{E}(u) + \gamma \cdot \|u\|_2^2, \quad \forall u \in \mathcal{D}. \tag{1.1}$$

Here $\tilde{u}$ denotes a quasi-continuous m-version of u, $\mathcal{E}(u) := \mathcal{E}(u,u)$, and $\|\cdot\|_2$ is the norm in $L^2(m)$. (In the terminology of [22], (1.1) amounts to the fact that the quadratic form $u \mapsto \int_E \tilde{u}^2\, d\mu$ (densely defined in $L^2(m)$) is $\mathcal{E}_1$-bounded.)

Decompose ν as the difference $\nu = \nu^+ - \nu^-$ of mutually singular smooth measures ν^+ and ν^-, and let C^+ (resp. C^-) be the PCAF corresponding to ν^+ (resp. ν^-). Let $|C|$ denote the total variation process of the signed CAF C. Then $|C|$ is a PCAF whose Revuz measure $|\nu|$ is

---

Date: November 5, 2002.

1991 *Mathematics Subject Classification.* Primary 31C25, secondary 60J57, 60J60, 60J45.

Key words and phrases. Dirichlet form, positivity preserving form, smooth measure, Kato class, Hardy class, Hardy inequality, local martingale, continuous additive functional, zero energy, even and odd continuous additive functionals, Fukushima decomposition, Nakao divergence, Feynman-Kac formula, Girsanov formula,

The second author was partially supported by a Grant-in-Aid for Scientific Research (C) No. 13640220 from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

(at least locally) the total variation of the signed measure ν. Our basic hypothesis is that each of the measures $\mu_{\langle M\rangle}$, $\mu_{\langle \hat{M}\rangle}$, $|\nu|$ is of the Hardy class, and that

$$\delta_0 := \sqrt{2\delta(M)} + \sqrt{2\delta(\hat{M})} + \delta(C^+) < 1, \tag{1.2}$$

where $\delta(M) = \delta(\mu_{\langle M\rangle})$ is as in (1.1) for the measure $\mu_{\langle M\rangle}$, etc. Under these conditions, we can define a quadratic form $\mathcal{Q}$ on $\mathcal{D}$ by

$$\begin{aligned} \mathcal{Q}(u,v) := \mathcal{E}(u,v) - \int_E \tilde{v}\, d\mu_{\langle M^{[u]},M\rangle} \\ - \int_E \tilde{u}\, d\mu_{\langle M^{[v]},\hat{M}\rangle} - \int_E \tilde{u}\tilde{v}\, d\nu, \qquad u,v\in\mathcal{D}, \end{aligned} \tag{1.3}$$

where $M^{[u]}, M^{[v]}$ are the square integrable martingale CAFs appearing in the Fukushima decomposition (2.5) below. (The covariation process $\langle M^{[u]}, M\rangle$ is a CAF with paths locally of finite variation and $\mu_{\langle M^{[u]},M\rangle}$ is the associated smooth signed measure, etc.) It is easy to check that there is a constant $k \in]0,\infty[$ such that

$$|\mathcal{Q}(u,v)| \le k\cdot \mathcal{E}_1(u)^{1/2}\mathcal{E}_1(v)^{1/2}, \qquad u,v\in\mathcal{D}. \tag{1.4}$$

Moreover,

$$\mathcal{Q}_\alpha(u,u) \ge (1-\delta_0)\mathcal{E}(u) + (\alpha-\alpha_0)\|u\|_2^2, \qquad u\in\mathcal{D}, \alpha>\alpha_0, \tag{1.5}$$

where $\mathcal{Q}_\alpha(u,v) := \mathcal{Q}(u,v) + \alpha(u,v)_m$ and $\alpha_0 := \gamma(M)\sqrt{2/\delta(M)} + \gamma(\hat{M})\sqrt{2/\delta(\hat{M})} + \gamma(C^+)$. The (non-symmetric) quadratic form $(\mathcal{Q},\mathcal{D})$ is a closed form on $L^2(m)$. Standard theory now yields the existence of a strongly continuous semigroup $(Q_t)_{t\ge 0}$ of bounded operators on $L^2(m)$ (with $\|Q_t\|_{2\to 2} \le e^{\alpha_0 t}$ for all $t\ge 0$) whose generator $(L^Q, D(L^{\mathcal{Q}}))$ is related to $\mathcal{Q}$ by

$$(-L^{\mathcal{Q}}u, v)_m = \mathcal{Q}(u,v), \quad \forall u\in D(L^{\mathcal{Q}}), v\in\mathcal{D}. \tag{1.6}$$

Notice that if $u\in\mathcal{D}$ then $u^+\in\mathcal{D}$ as well. Using the functional calculus for energy measures [17, Thm. 5.6.2] we obtain $\mu_{\langle M^{[u^+]}\rangle} = I_{\{\tilde{u}>0\}}\mu_{\langle M^{[u]}\rangle}$ for $u\in\mathcal{D}$; the estimate (1.1) now makes it is easy to see that for $\alpha>\alpha_0$,

$$\mathcal{Q}_\alpha(u^+,u) \ge 0, \quad \forall u\in\mathcal{D}. \tag{1.7}$$

It follows that the semigroup $(Q_t)_{t\ge 0}$ is *positivity preserving*: If $u\ge 0$ is an element of $L^2(m)$, then $Q_t u \ge 0$ m-a.e. This assertion also follows immediately from Theorem 1.1 below.

The statement of our main result requires the time reversal operators $r_t, t>0$, acting on the sample space Ω by $[r_t\omega](s) := \omega(t-s), 0\le s\le t$.

Theorem 1.1. *Define a multiplicative functional Z by*

$$Z_t := \exp[M_t + \hat{M}_t\circ r_t - \tfrac{1}{2}\langle M-\hat{M}\rangle_t + C_t], \quad t\ge 0, \tag{1.8}$$

and a family $(\overline{P}_t)_{t\ge 0}$ of operators by

$$\overline{P}_t f(x) := \mathbf{E}^x[Z_t f(X_t)]. \tag{1.9}$$

Then

(a) *$\overline{P}_t$ is a well-defined mapping of $L^2(m)$ into itself, and $(\overline{P}_t)_{t\ge 0}$ is a strongly continuous semigroup of bounded operators on $L^2(m)$, with $\|\overline{P}_t\|_{2\to 2} \le e^{\alpha_0 t}$ for $t>0$.*

(b) *For each $f\in L^2(m)$, $Q_t f = \overline{P}_t f$ m-a.e.*

It is clear from (1.3) that the adjoint semigroup (Q_t^*), which is associated with the quadratic form $\mathcal{Q}^*(u,v) := \mathcal{Q}(v,u)$, is the perturbation of (P_t) corresponding to the triple $(\hat{M}, M, C)$. This is consistent with Theorem 1.1, because (2.13) below implies that the adjoint $\overline{P}_t^*$ of $\overline{P}_t$ satisfies (1.9) with Z_t replaced by $Z_t \circ r_t$; in view of (1.8) and Proposition 2.1(c), the functional $Z_t \circ r_t$ satisfies (1.8) with the roles of M and $\hat{M}$ reversed.

Theorem 1.1 improves all results of its type of which we are aware, including [21, Prop. 3.4], [30, Thm. 3.4] and [43, Thm. 6.1], [44, Thm. 2.3], [47, Thm. II.6.2], and [52, Thm. 5.1]. We impose no structural assumptions on the diffusion $\mathbf{X}$, and condition (1.2) is weaker than conditions imposed by earlier authors. (It is customary in this area to work under conditions implying that $\mu_{\langle M\rangle}, \mu_{\langle \hat{M}\rangle}, |\nu|$ are measures of the Kato class.) Our method is a refinement of that used in [30], made possible by the use of even and odd CAFs [12] and Nakao's divergence [37].

Example 1.1. Suppose that $\mathbf{X}$ is standard Brownian motion in $\mathbb{R}^d$. The local martingale CAF M then admits a stochastic integral representation

$$M_t = \int_0^t b(X_s)\, dX_s, \tag{1.10}$$

where the measurable vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is pathwise locally square integrable, in the sense that

$$\int_0^t |b(X_s)|^2\, ds < \infty, \quad \forall t > 0, \tag{1.11}$$

$\mathbf{P}^m$-a.s. A similar representation, with vector field $\hat{b}$, holds for $\hat{M}$. Suppose, for simplicity, that $C_t := \int_0^t c(X_s)\, ds$ for some $c : \mathbb{R}^d \to \mathbb{R}$. A formal integration by parts in (1.3) shows that the infinitesimal generator $L^{\mathcal{Q}}$ can be expressed as

$$\begin{aligned} L^{\mathcal{Q}} u &= \tfrac{1}{2}\Delta u + b \cdot \nabla u - \operatorname{div}(u \cdot \hat{b}) + cu \\ &= \tfrac{1}{2}\Delta u + (b - \hat{b}) \cdot \nabla u + (c - \operatorname{div} \hat{b})u. \end{aligned} \tag{1.12}$$

The perturbation implied by the first expression for $L^{\mathcal{Q}}$ in (1.12) is clearly reflected in the formula (1.8) for Z_t. The second expression for $L^{\mathcal{Q}}$ in (1.12) is of technical importance in the proof of Theorem 1.1; the work of Nakao [37] provides a version of this expression that is valid in the general case.

Under condition (1.2) the semigroup defined by (1.9) operates in $L^p(m)$ for an interval $[p_-, p_+] \subset [1, \infty]$ of values of p. In the context of elliptic operators in $\mathbb{R}^d$, special cases of this result can be found in [5] ($M = \hat{M} = 0$) and [27] ($\hat{M} = 0$); see also [28], [23]. In the same context, a further refinement of the result presented below can be found in [48] and [29].

For $p \in [1, \infty]$ with conjugate exponent $q := p/(p-1)$ define

$$\delta_0(p) := \tfrac{2}{p}\sqrt{2\delta(M)} + \tfrac{2}{q}\sqrt{2\delta(\hat{M})} + \delta(C^+) \tag{1.13}$$

and

$$\alpha_0(p) := \tfrac{2}{p}\gamma(M)\sqrt{2/\delta(M)} + \tfrac{2}{q}\gamma(\hat{M})\sqrt{2/\delta(\hat{M})} + \gamma(C^+), \tag{1.14}$$

with the understanding that $\frac{2}{\infty} = 0$. Notice that $\delta_0 = \delta_0(2)$ and $\alpha_0 = \alpha_0(2)$. Now define

$$p_\pm := \frac{4}{2 - \kappa + \hat{\kappa} \mp \sqrt{(\kappa - \hat{\kappa} - 2)^2 - 4(2\hat{\kappa} + \lambda)}}, \tag{1.15}$$

where $\kappa := \sqrt{2\delta(M)}$, $\hat{\kappa} := \sqrt{2\delta(\hat{M})}$ and $\lambda := \delta(C^+)$. It is easy to check that $p_- < 2 < p_+$ provided $\delta_0(2) < 1$. Moreover, $p \in [p_-, p_+]$ if and only if $\delta_0(p) \leq \frac{2}{p} \cdot \frac{2}{q}$.

Theorem 1.2. *As before, suppose that $\delta_0(2) < 1$. If $p \in [p_-, p_+]$ then $\overline{P}_t$ maps $L^p(m)$ into itself and*

$$\|\overline{P}_t\|_{p \to p} \leq e^{\alpha_0(p)t}. \tag{1.16}$$

Furthermore, if p is finite then $(\overline{P}_t)_{t \geq 0}$ is a strongly continuous semigroup on $L^p(m)$.

In section 2 we describe precisely the setting in which we shall be working, and we recall necessary facts about symmetric diffusions and their Dirichlet forms. Section 3 contains the proof of Theorem 1.1 while section 4 contains the proof of Theorem 1.2. A number of examples are discussed in section 5.

2. Preliminaries

As stated in the first section, our interest is in Girsanov/Feynman-Kac transformations of a symmetric Markov diffusion process $\mathbf{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbf{P}^x)$. For general background on such diffusions we refer the reader to [17].

Getting down to specifies, we assume that the state space $(E, \mathcal{B}(E))$ of $\mathbf{X}$ is homeomorphic to a Borel subset of some compact metric space; here $\mathcal{B}(E)$ is the class of Borel subsets of E. Furthermore, we assume that the transition semigroup of $\mathbf{X}$, defined by

$$P_t f(x) := \mathbf{E}^x[f(X_t)], \qquad t \geq 0,\ f \in b\mathcal{B}(E), \tag{2.1}$$

maps $b\mathcal{B}(E)$ (the class of bounded real-valued $\mathcal{B}(E)$-measurable functions on E) into itself. In view of the symmetry of $\mathbf{X}$ (discussed below) this assumption of Borel measurability entails no loss of generality; see [16, §3]. To allow for the possibility that $P_t I_E(x) < 1$ for some states $x \in E$, we adjoin a cemetery state Δ to E; the process $\mathbf{X}$ is sent to Δ at its lifetime ζ. By convention any function (resp. measure) defined on E (resp. $\mathcal{E}^*$) is extended to Δ by declaring its value at Δ (resp. $\{\Delta\}$) to be 0.

By *diffusion* we mean that $\mathbf{X}$ is a strong Markov process such that (i) the paths $[0, \zeta(\omega)[\ni t \mapsto X_t(\omega)$ are continuous, and (ii) the lifetime ζ is a predictable stopping time. For this reason we can (and do) take the sample space Ω to be the space of paths ω from $[0, \infty[$ to $E \cup \{\Delta\}$ that are E-valued and continuous on $[0, \zeta(\omega)[$ and hold the value Δ after $\zeta(\omega) := \inf\{t | \, \omega(t) = \Delta\}$.

In saying that $\mathbf{X}$ is symmetric, with symmetry measure m, we mean that m is a σ-finite measure on $(E, \mathcal{B}(E))$ such that

$$(f, P_t g)_m = (P_t f, g)_m \qquad \forall f, g \in p\mathcal{B}(E), \tag{2.2}$$

where $(u, v)_m := \int_E uv \, dm$. In this case (P_t) restricted to $b\mathcal{B}(E) \cap L^2(m)$ extends uniquely to a strongly continuous contraction semigroup in $L^2(m)$. Of course, (2.2) implies that this semigroup is self-adjoint. The *Dirichlet form* associated with $\mathbf{X}$ is the bilinear form

$$\mathcal{E}(u, v) := \lim_{t \downarrow 0} t^{-1}(u, v - P_t v)_m \tag{2.3}$$

defined on the vector space

$$\mathcal{D} := \{u \in L^2(m) \mid \sup_{t > 0} t^{-1}(u, u - P_t u)_m < \infty\}. \tag{2.4}$$

From [15, §2] and the work of Albeverio, Ma, and Röckner [3, 4] we know that the Dirichlet form $(\mathcal{E}, \mathcal{D})$ is *quasi-regular*. See [33] and [17] for discussions of quasi-regularity. Because $\mathbf{X}$ is a diffusion, the Dirichlet form $(\mathcal{E}, \mathcal{D})$ is *strongly local*, in the sense that $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathcal{D}$ and u is constant m-a.e. on a neighborhood of the support of the measure $|v| \cdot m$; see [17] or [25].

The first exit time $\inf\{t > 0 \mid X_t \notin B\}$ of a Borel subset of E is denoted $\tau(B)$. An increasing sequence $\{B_n\}$ of Borel subsets of E is a *nest* provided $\mathbf{P}^m(\lim_n \tau(B_n) < \zeta) = 0$. The reader is referred to [33, Lemma IV. 4.5] for a characterization of this notion in terms of the Dirichlet form $(\mathcal{E}, \mathcal{D})$. A Borel set $N \subset E$ is said to be $\mathbf{X}$-*exceptional* provided $\mathbf{P}^m(\tau(E \setminus N) < \zeta) = 0$. It can be shown that $N \in \mathcal{B}(E)$ is $\mathbf{X}$-exceptional if and only if there is a nest of compacts $\{K_n\}$ such that $N \subset \cap_n K_n^c$.

Each element $u \in \mathcal{D}$ admits an m-modification $\tilde{u}$ such that $t \mapsto \tilde{u}(X_t)$ is continuous on $[0, \infty[$, $\mathbf{P}^m$-a.s. The function $\tilde{u}$ is *quasi-continuous* in the sense that there is a nest (K_n) of compact subsets of E such that $\tilde{u}|_{K_n} \in C(K_n)$ for every n. Moreover, the CAF $A_t^{[u]} := \tilde{u}(X_t) - \tilde{u}(X_0)$ admits a *Fukushima decomposition*

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \qquad \forall t \geq 0, \tag{2.5}$$

$\mathbf{P}^x$-a.s. for q.e. $x \in E$, in which $M^{[u]}$ and $N^{[u]}$ are CAFs of $\mathbf{X}$, $M^{[u]}$ is a martingale such that $\sup_{t>0} t^{-1}\mathbf{E}^m[(M^{[u]})^2] < \infty$, and $\lim_{t\to 0} \mathbf{E}^m[(N^{[u]})^2] = 0$. This decomposition is unique, and its components $M^{[u]}$ and $N^{[u]}$ are referred to as the martingale and zero-energy parts, respectively, of $A^{[u]}$.

We use $\mathcal{M}$ (resp. $\mathcal{N}$) to denote the class of martingale CAFs of finite energy (resp. CAFs of zero energy). To define these classes precisely, introduce the mutual energy $\mathbf{e}(M, N)$ of two CAFs by

$$\mathbf{e}(M, N) := \lim_{t\downarrow 0}(2t)^{-1}\mathbf{E}^m[M_t N_t] \tag{2.6}$$

whenever the limits exists. Then, writing $\mathbf{e}(M)$ for $\mathbf{e}(M, M)$,

$$\begin{aligned}\mathcal{M} := \{M \mid M \text{ is a CAF of } \mathbf{X};\ &\text{for each } t > 0,\ \mathbf{E}^x[M_t^2] < \infty \text{ and} \\ &\mathbf{E}^x[M_t] = 0 \text{ for q.e. } x \in E;\ \mathbf{e}(M) < \infty\},\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\mathcal{N} := \{N \mid N \text{ is a CAF of } \mathbf{X};\ &\text{for each } t > 0,\ \mathbf{E}^x[|N_t|] < \infty \\ &\text{for q.e. } x \in E;\ \mathbf{e}(N) = 0\}.\end{aligned} \tag{2.8}$$

When equipped with the inner product $(M, M') \mapsto \mathbf{e}(M, M')$, the class $\mathcal{M}$ is a Hilbert space.

A function $u : E \to \mathbb{R}$ is locally in $\mathcal{D}$ (notation: $u \in \mathcal{D}_{\text{loc}}$) provided there is a nest $\{E_n\}$ of finely open sets and a sequence $\{u_n\}$ of elements of $\mathcal{D}$ such that $u = u_n$, m-a.e. on E_n for each $n \in \mathbb{N}$. Each $u \in \mathcal{D}_{\text{loc}}$ admits a quasi-continuous modification $\tilde{u}$ such that (2.5) holds for $t \in [0, \zeta[$, where now $M^{[u]} \in \mathcal{M}_{\text{loc}}$ and $N^{[u]} \in \mathcal{N}_{\text{loc}}$. [The localizations $\mathcal{M}_{\text{loc}}$ and $\mathcal{N}_{\text{loc}}$ of $\mathcal{M}$ and $\mathcal{N}$ are defined in the obvious way; for example, $N \in \mathcal{N}_{\text{loc}}$ if and only if there is a nest $\{E_n\}$ of finely open sets and a sequence $\{N^{(n)}\}$ of elements of $\mathcal{N}$ such that $N_t = N_t^{(n)}$ for all $0 \leq t < \tau(E_n)$, $\mathbf{P}^m$-a.s. for each $n \in \mathbb{N}$. The elements of $\mathcal{M}_{\text{loc}}$ and $\mathcal{N}_{\text{loc}}$ are *local additive functionals* as discussed on page 226 in [17].]

If $M \in \mathcal{M}_{\text{loc}}$, then the quadratic variation process $\langle M \rangle$ is a PCAF of $\mathbf{X}$. The associated Revuz measure $\mu_{\langle M \rangle}$ is a smooth measure, and $M \in \mathcal{M}$ if and only if $\mu_{\langle M \rangle}(E) < \infty$, in which case $\mathbf{e}(M) = \frac{1}{2}\mu_{\langle M \rangle}(E)$. As a matter of notation, $\mu_{\langle u \rangle} := \mu_{\langle M^{[u]} \rangle}$ for $u \in \mathcal{D}_{\text{loc}}$. More generally, given $M, M' \in \mathcal{M}_{\rm loc}$, the covariation process $\langle M, M'\rangle$ is a CAF of $\mathbf{X}$ locally of finite variation; as such it admits a smooth signed Revuz measure $\mu_{\langle M,M'\rangle}$, and if M and M' both lie in $\mathcal{M}$ then $\mathbf{e}(M,M') = \frac{1}{2}\mu_{\langle M,M'\rangle}(E)$. If f is a quasi-continuous function, then the Itô integral

$$(f \bullet M)_t := \int_0^t f(X_s)\,dM_s \tag{2.9}$$

defines an element of $\mathcal{M}_{\rm loc}$. Evidently $f \bullet M \in \mathcal{M}$ if and only if $f \in L^2(\mu_{\langle M\rangle})$, in which case $\mathbf{e}(f \bullet M) = \frac{1}{2}\int_E f^2\,d\mu_{\langle M\rangle}$. In consonance with the Cauchy-Schwarz inequality satisfied by $\mathbf{e}$, we have, by the Kunita-Watanabe inequality [11, VII. 54],

$$\left|\int_E fg\,d\mu_{\langle M,M'\rangle}\right| \le \left(\int_E f^2\,d\mu_{\langle M\rangle}\right)^{1/2}\left(\int_E g^2\,d\mu_{\langle M'\rangle}\right)^{1/2}. \tag{2.10}$$

Also, because $\mu_{\langle M\pm M'\rangle} = \mu_{\langle M\rangle} \pm 2\mu_{\langle M,M'\rangle} + \mu_{\langle M'\rangle}$ is a positive measure, we have

$$\mu_{\langle M+M'\rangle} \le 2\mu_{\langle M\rangle} + 2\mu_{\langle M'\rangle}. \tag{2.11}$$

Given $t > 0$ and a path ω with $\zeta(\omega) > t$, define the reversed (at time t) path $r_t\omega$ by

$$r_t(\omega)(s) := \begin{cases} \omega(t-s), & 0 \le s \le t, \\ \omega(0), & s > t. \end{cases} \tag{2.12}$$

Actually, r_t is a mapping of equivalence classes of paths that coincide up to time t; thus $F \circ r_t$ is well defined on $\{t < \zeta\}$ provided F is $\mathcal{F}_t$-measurable. See [50] for more details. The importance of r_t stems from the following amplification of the symmetry property (2.2): For every $t > 0$ and every positive $\mathcal{F}_t$-measurable function F,

$$\mathbf{E}^m[F \circ r_t : t < \zeta] = \mathbf{E}^m[F : t < \zeta]. \tag{2.13}$$

Given a CAF A of $\mathbf{X}$, the *even* and *odd* parts of A are defined by the formulae

$$A_t^{\rm even} := [A_t + A_t \circ r_t]/2, \qquad A_t^{\rm odd} := [A_t - A_t \circ r_t]/2, \qquad 0 \le t < \zeta. \tag{2.14}$$

We say that A is even (resp. odd) provided $A_t = A_t^{\rm even}$ (resp. $A_t = A_t^{\rm odd}$) $\mathbf{P}^m$-a.s. on $\{t < \zeta\}$ for each $t > 0$. It is easy to check to that $A_{t+s}^{\rm even} = A_t^{\rm even} + A_s^{\rm even} \circ \theta_t$ $\mathbf{P}^m$-a.s. on $\{t + s < \zeta\}$, and similarly for $A^{\rm odd}$.

For proofs of the facts asserted in the following proposition, see [12, 13].

Proposition 2.1. (a) *Every element of $\mathcal{N}_{\rm loc}$ is an even CAF.*
- (b) *If $u \in \mathcal{D}_{\rm loc}$, then the even part of the local martingale CAF $M^{[u]}$ is $-N^{[u]}$, while the odd part is $A^{[u]}$.*
- (c) *Every CAF locally of finite variation is even.*

The odd part of a general martingale CAF can be expressed in terms of a divergence-like functional defined by S. Nakao: according to [37, Section 3], there is a linear map $\Lambda : \mathcal{M} \to \mathcal{N}$, such that

$$\lim_{t\downarrow 0} t^{-1}\mathbf{E}^m[h(X_0)\Lambda(M)_t] = -\mathbf{e}(M^{[h]}, M), \qquad \forall M \in \mathcal{M},\ h \in \mathcal{D}. \tag{2.15}$$

In particular, $\Lambda(M^{[u]}) = N^{[u]}$ for all $u \in \mathcal{D}$, because of [17, Thm. 5.2.4]. By [37, Thm. 2.2], the identity (2.15) uniquely determines $\Lambda(M) \in \mathcal{N}$. Localization arguments allow one to extend Λ to a linear mapping of $\mathcal{M}_{\rm loc}$ into $\mathcal{N}_{\rm loc}$ such that $M^{\rm odd} = M + \Lambda(M)$ and $M^{\rm even} = -\Lambda(M)$ for all $M \in \mathcal{M}_{\rm loc}$. It should be noted that $\Lambda(M) \in \mathcal{N}_{\rm loc}$ for $M \in \mathcal{M}_{\rm loc}$ is uniquely determined in the sense that for $M^{(1)}, M^{(2)} \in \mathcal{M}_{\mathrm{loc}}$, $M_t^{(1)} = M_t^{(2)}$ $\forall t \in [0, \zeta[$ $\mathbf{P}^x$-a.s. for q.e. $x \in E$ implies $\Lambda(M^{(1)})_t = \Lambda(M^{(2)})_t$ $\forall t \in [0, \zeta[$ $\mathbf{P}^x$-a.s. for q.e. $x \in E$.

A (positive) measure μ on E is said to be of the *Kato class* (notation $\mu \in S_K$) provided an estimate of the form

$$|\mu U_\alpha(f)| \le c_\alpha \|f\|_1, \qquad \forall f \in L^1(m), \tag{2.16}$$

obtains, with $\lim_{\alpha\to\infty} c_\alpha = 0$. If (2.16) holds for even a single $\alpha > 0$ with $c_\alpha \in [0, \infty[$, then μ is said to be of *Dynkin class* (notation $\mu \in S_D$). Any element of S_D is a smooth measure, by [16, (4.15)]. Alternatively, a smooth measure μ, with associated PCAF A^μ, is of the Kato class (resp. Dynkin class) if and only if the function $a_t(x) := \mathbf{E}^x[A_t^\mu]$ satisfies

$$\lim_{t\to 0} \|a_t\|_\infty = 0, \quad (\text{resp. } a_t \in L^\infty(m) \text{ for some (or all) } t > 0), \tag{2.17}$$

where the norm $\|\cdot\|_\infty$ is that of $L^\infty(m)$.

Given $\mu \in S_D$, [14, Example (1.17)] implies that (1.1) holds with $\delta = c_\alpha$ and $\gamma = \alpha c_\alpha$; *cf.* [22, Lemma VI. 4.8a] and [49, Thm. 3.1]. Thus $S_K \subset S_D \subset S_H$, where the Hardy class S_H is as defined in section 1.

By combining [49, Proposition 1.9] with the proof of [14, Thm. 4.22], we obtain the following link between S_K and the class of smooth measures.

Proposition 2.2. *If μ is a positive smooth measure, then there exists a strictly positive quasi-continuous $\varphi \in \mathcal{D}$ such that the measure $\varphi \cdot \mu$ is of the Kato class.*

3. Proof of Theorem 1.1

We now fix a triple $(M, \hat{M}, C)$ as described in section 1; especially, condition (1.2) holds. The bilinear form $\mathcal{Q}$, its associated semigroup (Q_t), and the multiplicative functional Z are also as in section 1. Let $(V_\alpha)_{\alpha > \alpha_0}$ be the resolvent associated with $(\mathcal{Q}, \mathcal{D})$. It follows immediately from (1.4) and (1.5) that $(\mathcal{Q}, \mathcal{D})$ is a quasi-regular positivity preserving form, and that the quasi notions (nest, quasi-continuity, etc.) associated with $(\mathcal{Q}, \mathcal{D})$ coincide with the analogous notions for $(\mathcal{E}, \mathcal{D})$. The same remarks apply to the restricted forms $(\mathcal{Q}^{(n)}, \mathcal{D}^{(n)})$ considered below.

Before embarking on the proof of Theorem 1.1, we establish some properties of Z. To this end, the following expression for Z_t will be useful. Because $\Lambda(\hat{M})$ is even and $\hat{M} + \Lambda(\hat{M})$ is odd,

$$\hat{M}_t \circ r_t = [\hat{M}_t + \Lambda(\hat{M})_t] \circ r_t - \Lambda(\hat{M})_t = -\hat{M}_t - 2\Lambda(\hat{M})_t \tag{3.1}$$

$\mathbf{P}^m$-a.s. on $\{t < \zeta\}$ for each $t > 0$. Therefore

$$Z_t = \exp\left(K_t - \tfrac{1}{2}\langle K\rangle_t + C_t - 2\Lambda(\hat{M})_t\right), \quad \mathbf{P}^m\text{-a.s. on } \{t < \zeta\}, \quad \forall t > 0, \tag{3.2}$$

where $K := M - \hat{M}$.

With regard to condition (ii) in part (b) of the lemma to follow, notice that if $\hat{M} = M^o + M^{[u]}$ where M^o is odd, then $\Lambda(\hat{M}) = N^{[u]}$. For general $\hat{M}$, the representation $\Lambda(\hat{M}) = N^{[u]}$ is locally true, by an important result of Ōshima and Yamada [41].

Lemma 3.1. (a) *If $\mu_{\langle M\rangle} + \mu_{\langle \hat{M}\rangle} + |\nu| \in S_K$, then $\overline{P}_t$ defined by (1.9) maps $L^2(m)$ into itself and there is a constant c_1 such that $\|\overline{P}_t\|_{2\to 2} \le c_1 \cdot e^{c_1 t}$ for all $t > 0$.*

(b) *Suppose that (i)* $\mu_{\langle M\rangle}+\mu_{\langle \hat M\rangle}+|\nu|\in S_K$*, (ii)* $2\Lambda(\hat M)=N^{[\rho]}$ *for some* $\rho\in\mathcal{D}_{\rm loc}$*, (iii)* $\mu_{\langle\rho\rangle}\in S_K$*, and (iv)* $\tilde\rho$ *is bounded. Then there is a constant* c_2 *such that for* m*-a.e.* $x\in E$*,*

$$\mathbf{E}^x\left[\sup_{0\le s<t\wedge\zeta} Z_s\right]\le c_2e^{c_2t},\qquad \forall t>0. \tag{3.3}$$

Proof. (a) Decompose the CAF C as the difference C^+-C^- of PCAFs with mutually singular Revuz measures ν^+ and ν^-, respectively. Then $\nu^+\le|\nu|$, and because $\mu_{\langle M\rangle}+\mu_{\langle \hat M\rangle}+|\nu|\in S_K$, so also

$$\eta:=(9/2)[\mu_{\langle M\rangle}+\mu_{\langle \hat M\rangle}]+3\nu^+\in S_K. \tag{3.4}$$

Let f and g be non-negative elements of $L^2(m)$. Then by Hölder's inequality and the expression (1.8) for Z_t,

$$\begin{aligned}\mathbf{E}^m[f(X_0)Z_tg(X_t)]\le{}&\mathbf{E}^m[f(X_0)^2e^{3M_t-(9/2)\langle M\rangle_t}]^{1/3}\\&\times\mathbf{E}^m[g(X_t)^2e^{3\hat M_t\circ r_t-(9/2)\langle\hat M\rangle_t}]^{1/3}\mathbf{E}^m[f(X_0)e^{B_t}g(X_t)]^{1/3},\end{aligned} \tag{3.5}$$

where $B_t:=(9/2)\langle M\rangle_t+(9/2)\langle\hat M\rangle_t+3C_t^+$ is the PCAF with Revuz measure η. Now $e^{3M_t-(9/2)\langle M\rangle_t}$ is a positive supermartingale, so $\mathbf{E}^x[e^{3M_t-(9/2)\langle M\rangle_t}]\le 1$ for q.e. $x\in E$. Thus the first factor on the right side of (3.5) is no bigger than $\|f\|_2^{2/3}$. Because $\langle\hat M\rangle$ is even, the middle factor on the right side of (3.5) is equal to

$$\mathbf{E}^m[g(X_t)^2e^{3\hat M_t\circ r_t-(9/2)\langle\hat M\rangle_t\circ r_t}]^{1/3}=\mathbf{E}^m[g(X_0)^2e^{3\hat M_t-(9/2)\langle\hat M\rangle_t}]^{1/3}\le\|g\|_2^{2/3}, \tag{3.6}$$

because $e^{3\hat M_t-(9/2)\langle\hat M\rangle_t}$ is also a positive supermartingale. Finally, by [19, (3.14)], (3.4) implies the existence of a constant c_1 such that the third factor on the right side of (3.5) is no bigger than $c_1e^{c_1t}\|f\|_2^{1/3}\|g\|_2^{1/3}$. Combining these estimates we find that

$$\mathbf{E}^m[f(X_0)Z_tg(X_t)]\le c_1\cdot e^{c_1t}\|f\|_2\|g\|_2, \tag{3.7}$$

which proves the assertion.

(b) By hypothesis there exists $r>0$ such that $|\tilde\rho|\le r$ for q.e. $x\in E$. Using (3.2) and the notation C^+ from the proof of part (a), we see that

$$\sup_{0\le s<t\wedge\zeta}Z_s\le e^{2r}\sup_{0\le s\le t}\exp\left(J_s-\tfrac12\langle J\rangle_s\right)\exp\left(\tfrac12\langle J\rangle_t+C_t^+\right), \tag{3.8}$$

$\mathbf{P}^m$-a.s., where $J:=K+M^{[\rho]}$. Now $\mu_{\langle J\rangle}\le 4\mu_{\langle M\rangle}+4\mu_{\langle\hat M\rangle}+2\mu_{\langle\rho\rangle}$ by (2.11), so the Revuz measures of $\langle J\rangle$ and C^+ are of the Kato class; thus there is a constant $k\in]0,\infty[$ such that for q.e. $x\in E$,

$$\mathbf{E}^x[\exp(\langle J\rangle_t+2C_t^+)]\le ke^{kt} \tag{3.9}$$

and

$$\mathbf{E}^x[\exp(6\langle J\rangle_t)]\le ke^{kt} \tag{3.10}$$

for all $t>0$; see for example [16, (3.11)]. Applying Doob's inequality to the martingale $\exp(J-\frac12\langle J\rangle)$ we obtain

$$\begin{aligned}\mathbf{E}^x\left[\left|\sup_{0\le s\le t}\exp(J_s-\tfrac12\langle J\rangle_s)\right|^2\right]&\le 4\mathbf{E}^x[\exp(2J_t-\langle J\rangle_t)]\\&\le 4\mathbf{E}^x[\exp(4J_t-8\langle J\rangle_t)]^{1/2}\mathbf{E}^x[\exp(6\langle J\rangle_t)]^{1/2}\\&\le 4\sqrt{k}e^{(k/2)t},\end{aligned} \tag{3.11}$$

because $\exp(4J_t - 8\langle J\rangle_t)$ is a martingale. In view of (3.9) and (3.11), the assertion follows from (3.8) and the Cauchy-Schwarz inequality. $\square$

Let $L^{\mathcal{Q}}$, with domain $D(L^{\mathcal{Q}})$, denote the infinitesimal generator of the semigroup (Q_t). Fix $u \in D(L^{\mathcal{Q}})$. Then by (1.3), for all $v \in \mathcal{D}$

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{Q}(u,v) + \int_E \tilde v\, d\mu_{\langle M^{[u]},M\rangle} + \int_E \tilde u\, d\mu_{\langle M^{[v]},\hat M\rangle} + \int_E \tilde u\tilde v\, d\nu \\ &= (-L^{\mathcal{Q}}u,v)_m + \int_E \tilde v\, d\mu_{\langle M^{[u]},M-\hat M\rangle} \\ &\qquad +2\mathbf{e}(\tilde u\bullet M^{[v]} + \tilde v \bullet M^{[u]}, \hat M) + \int_E \tilde u\tilde v\, d\nu.\end{aligned} \tag{3.12}$$

From this it follows that

$$\begin{aligned}N_t^{[u]} &= \int_0^t L^{\mathcal{Q}}u(X_s)\, ds - \langle M^{[u]},K\rangle_t \\ &\qquad +2\int_0^t \tilde u(X_s)\circ\, d\Lambda(\hat M)_s - \int_0^t \tilde u(X_s)\, dC_s,\end{aligned} \tag{3.13}$$

where the Stratonovich-type integral $\int_0^t \tilde u(X_s)\circ\, d\Lambda(\hat M)_s$ is defined by

$$\int_0^t \tilde u(X_s)\circ d\Lambda(\hat M)_s := \Lambda\left(\int_0^\cdot \tilde u(X_s)\, d\hat M_s\right)_t - \tfrac12\langle M^{[u]},\hat M\rangle_t. \tag{3.14}$$

That is, the Fukushima decomposition associated with u is

$$\begin{aligned}\tilde u(X_t) - \tilde u(X_0) &= M_t^{[u]} + \int_0^t L^{\mathcal{Q}}u(X_s)\, ds - \langle K, M^{[u]}\rangle_t \\ &\qquad +2\int_0^t \tilde u(X_s)\circ d\Lambda(\hat M)_s - \int_0^t \tilde u(X_s)\, dC_s.\end{aligned} \tag{3.15}$$

Lemma 3.2. *Suppose, in addition to the other hypothesis of Theorem 1.1, that $2\Lambda(\hat M) = N^{[\rho]}$ for some $\rho\in\mathcal{D}_{\rm loc}$. Then for each $t>0$,*

$$\tilde u(X_t)Z_t = \tilde u(X_0) + \int_0^t Z_s\, dM_s^{[u]} + \int_0^t Z_s\tilde u(X_s)\, dK_s + \int_0^t Z_s L^{\mathcal{Q}}u(X_s)\, ds, \tag{3.16}$$

$\mathbf{P}^m$-a.s. on $\{t<\zeta\}$.

Proof. Because of the assumed form of $\Lambda(\hat M)$, we have $\hat M\circ r_t = -\hat M_t - N_t^{[\rho]}$. Using this and (3.2) we factor Z as

$$Z_t = \exp(K_t - \tfrac12\langle K\rangle_t + C_t + \tilde\rho(X_0) + M_t^{[\rho]})\exp(-\tilde\rho(X_t)),\ \mathbf{P}^m\text{-a.s. on } \{t<\zeta\}. \tag{3.17}$$

By [37, §4] or [31, Thm. 2.4.1] we have

$$\begin{aligned}&\tilde u(X_t)e^{-\tilde\rho(X_t)} - \tilde u(X_0)e^{-\tilde\rho(X_0)} \\ &\quad= -\int_0^t \tilde u(X_s)e^{-\tilde\rho(X_s)}\, dM_s^{[\rho]} + \int_0^t e^{-\tilde\rho(X_s)}\, dM_s^{[u]} \\ &\qquad -\int_0^t \tilde u(X_s)e^{-\tilde\rho(X_s)}\circ dN_s^{[\rho]} + \int_0^t e^{-\tilde\rho(X_s)}\circ dN_s^{[u]} \\ &\qquad +\tfrac12\int_0^t \tilde u(X_s)e^{-\tilde\rho(X_s)}\, d\langle M^{[\rho]}\rangle_s - \int_0^t e^{-\tilde\rho(X_s)}\, d\langle M^{[u]},M^{[\rho]}\rangle_s.\end{aligned} \tag{3.18}$$

We focus on the terms

$$-\int_0^t \tilde{u}(X_s)e^{-\tilde{\rho}(X_s)}\circ dN_s^{[\rho]} + \int_0^t e^{-\tilde{\rho}(X_s)}\circ dN_s^{[u]}. \tag{3.19}$$

In view of (3.13),

$$N_t^{[u]} = \int_0^t L^{\mathcal{Q}}u(X_s)\,ds - \langle M^{[u]}, K\rangle_t + \int_0^t \tilde{u}(X_s)\circ dN_s^{[\rho]} - \int_0^t \tilde{u}(X_s)\,dC_s. \tag{3.20}$$

By the injectivity of $\Lambda : \mathcal{M}_{\rm loc} \to \mathcal{N}_{\rm loc}$, we have

$$\int_0^t \tilde{u}(X_s)\circ dN_s^{[\rho]} = \Lambda\biggl(\int_0^\cdot \tilde{u}(X_s)\,dM_s^{[\rho]}\biggr)_t - \tfrac{1}{2}\langle M^{[u]}, M^{[\rho]}\rangle_t. \tag{3.21}$$

Consequently,

$$\begin{aligned}
&\int_0^t e^{-\tilde{\rho}(X_s)}\circ d\biggl(\int_0^\cdot \tilde{u}(X_r)\circ N_r^{[\rho]}\biggr)_s \\
&\quad= \int_0^t e^{-\tilde{\rho}(X_s)}\circ d\Lambda\biggl(\int_0^\cdot \tilde{u}(X_r)\,dM_r^{[\rho]}\biggr)_s - \tfrac{1}{2}\int_0^t e^{-\tilde{\rho}(X_s)}\,d\langle M^{[u]}, M^{[\rho]}\rangle_s \\
&\quad= \Lambda\biggl(\int_0^\cdot e^{-\tilde{\rho}(X_s)}\tilde{u}(X_s)\,dM_s^{[\rho]}\biggr)_t - \tfrac{1}{2}\langle M^{[e^{-\rho}]}, u\bullet M^{[\rho]}\rangle_t \\
&\qquad -\tfrac{1}{2}\int_0^t e^{-\tilde{\rho}(X_s)}\,d\langle M^{[u]}, M^{[\rho]}\rangle_s \\
&\quad= \Lambda\biggl(\int_0^\cdot e^{-\tilde{\rho}(X_s)}\tilde{u}(X_s)\,dM_s^{[\rho]}\biggr)_t - \tfrac{1}{2}\langle M^{[ue^{-\rho}]}, M^{[\rho]}\rangle_s \\
&\quad= \int_0^t [\tilde{u}(X_s)e^{-\tilde{\rho}(X_s)}]\circ dN_s^{[\rho]}.
\end{aligned} \tag{3.22}$$

Thus, (3.19) is equal to

$$\int_0^t e^{-\tilde{\rho}(X_s)}L^{\mathcal{Q}}u(X_s)\,ds - \int_0^t e^{-\tilde{\rho}(X_s)}\,d\langle K, M^{[u]}\rangle_s - \int_0^t e^{-\tilde{\rho}(X_s)}\tilde{u}(X_s)\,dC_s, \tag{3.23}$$

which is evidently a continuous semimartingale. Feeding (3.23) into (3.18), then using Itô's formula, we obtain (after a straightforward, if tedious, calculation) the assertion of the lemma. □

Proof of Theorem 1.1. We would like to take expectations of both sides of (3.16), but we must deal with the fact that the local martingale terms on the right side need not be martingales. To this end let $\{f_j\}_{j\in\mathbb{N}} \subset L^2(m)$ be total in $L^2(m)$ and such that $f_j \ge 0$ for each $j \in \mathbb{N}$. Fix $\alpha > \alpha_0$ and notice that the linear span of $\{V_\alpha f_j \mid j \in \mathbb{N}\}$ is contained in $D(L^{\mathcal{Q}})$ and is dense therein with respect to the graph norm: If $u \in D(L^{\mathcal{Q}})$ then there is a sequence $\{g_k\}$ of elements of the linear span of $\{f_j\}$ such that $\lim_k \|V_\alpha g_k - u\|_2 = 0$ and $\lim_k \|L^{\mathcal{Q}}(V_\alpha g_k) - L^{\mathcal{Q}}u\|_2 = 0$. Each $V_\alpha f_j$, being an element of $\mathcal{D}$, admits a quasi-continuous m-version $\widetilde{V_\alpha f_j}$; therefore, by a standard construction [17, Thm. 2.1.2], there is an $\mathcal{E}$-nest of compacts $\{K_n^1\}$ such that $\widetilde{V_\alpha f_j}|_{K_n^1} \in C(K_n^1)$ for each n and each j. Next, by (2.2), there is a strictly positive quasi-continuous function φ such that the measure $\varphi\cdot(\mu_{\langle M\rangle} + \mu_{\langle \hat{M}\rangle} + |\nu|)$ is an element of S_K. Because φ is quasi-continuous, there is a nest $\{K_n^2\}$ such that $\varphi|_{K_n^2} \in C(K_n^2)$ for each n. By [41, Thm. 2.1], there is a nest $\{K_n^3\}$ and a sequence $\rho_n \in \mathcal{D}_{\rm loc}$ such that $2\Lambda(\hat{M})_t = N_t^{[\rho_n]}$ for $\mathbf{P}^m$-a.s. on $0 \le t < \tau(K_n^3)$ for each n. Finally, since $\mu_{\langle \rho_n\rangle}$ is a smooth measure and $\tilde{\rho}_n$ is quasi-continuous, there is a nest $\{K_{n,j}\}_{j\in\mathbb{N}}$ such that $I_{K_{n,j}}\cdot \mu_{\langle\rho_n\rangle} \in S_K$ and $\tilde{\rho}_n|_{K_{n,j}} \in C(K_{n,j})$ for each $n, j \in \mathbb{N}$. Let π be a probability measure equivalent to m; then $\lim_j \mathbf{E}^\pi[\exp(-\tau(\hat{K}_n \cap K_{n,j})) - \exp(-\tau(\hat{K}_n))] = 0$ so we can choose j_n so large that $\mathbf{E}^\pi[\exp(-\tau(\hat{K}_n \cap K_{n,j_n})) - \exp(-\tau(\hat{K}_n))] < 1/n^2$, where $\hat{K}_n := K_n^1 \cap K_n^2 \cap K_n^3$, Clearly

$$K_n := \hat{K}_n \cap K_{n,j_n} \cap \{\varphi \geq 1/n\}, \qquad n \in \mathbb{N} \tag{3.24}$$

defines a nest of compact sets. Let E_n denote the fine interior of K_n with respect to $\mathbf{X}$. Then E_n is a finely open nearly Borel set and is quasi-open.

Define $\mathcal{D}^{(n)} := \{u \in \mathcal{D} \mid \tilde{u} = 0 \text{ q.e. on } E_n^c\}$, and let $\mathcal{Q}^{(n)}$ denote the restriction of $\mathcal{Q}$ to $\mathcal{D}^{(n)}$. Clearly $(\mathcal{Q}^{(n)}, \mathcal{D}^{(n)})$ is a quasi-regular positivity preserving coercive closed form on $L^2(E_n; m)$ satisfying the same hypothesis as $(\mathcal{Q}, \mathcal{D})$. In fact, $(\mathcal{Q}^{(n)}, \mathcal{D}^{(n)})$ is related to the restriction of $\mathcal{E}$ to $\mathcal{D}^{(n)}$ (which is the Dirichlet form of the part process $\mathbf{X}^{E_n}$) in exactly the same way that $(\mathcal{Q}, \mathcal{D})$ is related to $(\mathcal{E}, \mathcal{D})$. In addition, the conditions of Lemma 3.1 and Lemma 3.2 are satisfied by $\mathbf{X}^{E_n}$ and $(\mathcal{Q}^{(n)}, \mathcal{D}^{(n)})$. Let $(Q_t^{(n)})_{t>0}$ and $(V_\alpha^{(n)})_{\alpha>\alpha_0}$ be the semigroup and resolvent on $L^2(E_n; m)$ associated with $(\mathcal{Q}^{(n)}, \mathcal{D}^{(n)})$. Let $(L^{\mathcal{Q},(n)}, D(L^{\mathcal{Q},(n)}))$ denote the infinitesimal generator of $(Q_t^{(n)})$. Consider a bounded $u \in D(L^{\mathcal{Q},(n)})$. Let τ_n be the first exit time of $\mathbf{X}$ from E_n. Then by Lemma 3.2 (applied to $\mathbf{X}^{E_n}$ and $(\mathcal{Q}^{(n)}, \mathcal{D}^{(n)})$),

$$\begin{aligned}\tilde{u}(X_{t\wedge\tau_n})Z_{t\wedge\tau_n} = \tilde{u}(X_0) &+ \int_0^{t\wedge\tau_n} Z_s \, dM_s^{[u],(n)} \\ &+ \int_0^{t\wedge\tau_n} Z_s \tilde{u}(X_s)\, dK_s + \int_0^{t\wedge\tau_n} Z_s L^{\mathcal{Q},(n)} u(X_s)\, ds,\end{aligned} \tag{3.25}$$

because $u(X_{\tau_n}) = 0$, $\mathbf{P}^m$-a.s. Here was used the fact that $Z^{(n)}$ (resp. $K^{(n)}$), the analog of Z (resp. K) with respect to $\mathbf{X}^{E_n}$ and $(\mathcal{Q}^{(n)}, \mathcal{D}^{(n)})$, coincides with Z (resp. K) on $[0, \tau_n[$. Let $\{T_k\}$ be an increasing sequence of stopping times with $T_k \uparrow \zeta$ as $k \to \infty$ reducing the local martingale terms on the right hand side of (3.25). Replacing t by $t \wedge T_k$ in (3.25) and taking expectations we obtain

$$\mathbf{E}^x[\tilde{u}(X_{t\wedge T_k\wedge\tau_n})Z_{t\wedge T_k\wedge\tau_n}] = \tilde{u}(x) + \mathbf{E}^x\left[\int_0^{t\wedge T_k\wedge\tau_n} Z_s (L^{\mathcal{Q},(n)}u)(X_s)\, ds\right], \tag{3.26}$$

for m-a.e. $x \in E_n$. Because u is bounded, Lemma 3.1(b) permits us to conclude that, as $k \to \infty$, the left side of (3.26) converges to $\mathbf{E}^x[\tilde{u}(X_{t\wedge\tau_n})]$ for m-a.e. $x \in E_n$. On the other hand

$$\left|\mathbf{E}^x\left[\int_0^{t\wedge\tau_n} Z_s (L^{\mathcal{Q},(n)}u)(X_s)\, ds\right]\right| \leq \int_0^t \overline{P}_s^{(n)} |L^{\mathcal{Q},(n)}u|(x)\, ds < \infty \tag{3.27}$$

for m-a.e. $x \in E_n$, where

$$\overline{P}_s^{(n)} f(x) := \mathbf{E}^x[f(X_s)Z_s : s < \tau_n] \tag{3.28}$$

and the final inequality on the right hand side of (3.27) follows from Lemma 3.1(a). Passing to the limit in (3.25) as $k \to \infty$, we obtain

$$\mathbf{E}^x[\tilde{u}(X_{t\wedge\tau_n})Z_{t\wedge\tau_n}] = u(x) + \mathbf{E}^x\left[\int_0^{t\wedge\tau_n} Z_s (L^{\mathcal{Q},(n)}u)(X_s)\, ds\right] \tag{3.29}$$

for m-a.e. $x \in E_n$, provided $u \in D(L^{\mathcal{Q},(n)})$ is bounded. For $u \in D(L^{\mathcal{Q},(n)})$ of the form $V_\alpha^{(n)} g$ ($0 \leq g \in L^2(E_n; m)$), by the construction of the nest $\{K_n\}$, there is a sequence $\{g_k\}$ of non-negative elements of $L^2(m)$ such that $u_k := V_\alpha^{(n)} g_k$ is in $L^\infty(E_n; m)$, u_k converges in $L^2(E_n; m)$ to u and $L^{\mathcal{Q},(n)} u_k = \alpha u_k - g_k$ converges to $L^{\mathcal{Q},(n)} u$ in $L^2(E_n; m)$. [The boundedness of u_k comes from the inequality $V_\alpha^{(n)} f \le 1_{K_n} \cdot V_\alpha f$ for all non-negative $f \in L^2(m)$, which is a consequence of the fact that these resolvents are associated with quasi-regular positivity preserving forms. Indeed, the inequality is easy to prove if $(V_\alpha)_{\alpha>\alpha_0}$ is subMarkov, hence the resolvent of a right process; the h-transform technique of section 5 of [34] allows us to reduce the general case to the subMarkov case.] Substituting u_k for u in (3.29) and then passing the limit as $k \to \infty$, we see that (3.29) is valid for all $u \in D(L^{\mathcal{Q},(n)})$, since any such u can be written as $V_\alpha^{(n)} g_1 - V_\alpha^{(n)} g_2$ for non-negative $g_1, g_2 \in L^2(E_n; m)$. That is, we have,

$$\overline{P}_t^{(n)} u(x) = u(x) + \int_0^t \overline{P}_s^{(n)} (L^{\mathcal{Q},(n)} u)(x)\, ds, \qquad m\text{-a.e. } x \in E_n, \tag{3.30}$$

for all $u \in D(L^{\mathcal{Q},(n)})$. This implies the strong continuity of $\overline{P}_t^{(n)}$ on $D(L^{\mathcal{Q},(n)})$, hence on $L^2(E_n; m)$. Note that $\overline{P}_t^{(n)}$ maps $L^2(E_n; m)$ into itself by Lemma 3.1(a), because $I_{E_n}(\mu_{\langle M\rangle} + \mu_{\langle \hat{M}\rangle} + |\nu|)$ is a Kato class smooth measure with respect to $\mathbf{X}^{E_n}$. Hence

$$\lim_{t\downarrow 0} \frac{\overline{P}_t^{(n)} u - u}{t} = L^{\mathcal{Q},(n)} u. \tag{3.31}$$

Thus, using $(\overline{L}^{(n)}, D(\overline{L}^{(n)}))$ to denote the infinitesimal generator of $(\overline{P}_t^{(n)})_{t>0}$, we have shown

$$D(L^{\mathcal{Q},(n)}) \subset D(\overline{L}^{(n)}) \text{ and } L^{\mathcal{Q},(n)} = \overline{L}^{(n)} \text{ on } D(L^{\mathcal{Q},(n)}). \tag{3.32}$$

Let $\overline{U}_\alpha^{(n)} := \int_0^\infty e^{-\alpha t} \overline{P}_t^{(n)}\, dt$ denote the resolvent operator associated with $\overline{L}^{(n)}$. Fix $f \in L^2(E_n; m)$ and define, for $\alpha > \alpha_0$, $v := V_\alpha^{(n)} f$. Then $v \in D(L^{\mathcal{Q},(n)}) \subset D(\overline{L}^{(n)})$ and $\overline{L}^{(n)} v = L^{\mathcal{Q},(n)} v = \alpha v - f$. But $D(\overline{L}^{(n)})$ coincides with $\overline{U}_\alpha^{(n)}(L^2(E_n; m))$, so there exists $g \in L^2(E_n; m)$ such that $v = \overline{U}_\alpha^{(n)} g$, in which case $\overline{L}^{(n)} v = \alpha v - g$. It follows that $f = g$, and then that $\overline{U}_\alpha^{(n)} f = V_\alpha^{(n)} f$, for all $f \in L^2(E_n; m)$. This identity of resolvents implies that the associated semigroups $(\overline{P}_t^{(n)})$ and $(Q_t^{(n)})$ coincide.

It remains to let $n \to \infty$. Since $\lim_n \tau_n = \zeta$, $\mathbf{P}^m$-a.s., dominated convergence tells us that $\lim_n \overline{U}_\alpha^{(n)} f(x) = \overline{U}_\alpha f(x)$ for m-a.e. $x \in E$ and each $f \in L^2(m)$. On the other hand, by Lemma 3.6 in [25], $\lim_n V_\alpha^{(n)} f = V_\alpha f$ in a norm stronger than that of $L^2(m)$ for f in a dense subset of $L^2(m)$. It follows that the resolvents of the semigroups $(\overline{P}_t)$ and (Q_t) coincide, hence so do the semigroups. Our theorem is proved. □

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is adapted from [27] but the method is well known; see [9, § 2.2]. By duality it suffices to consider the case $p \in]2, p_+]$. Fix a non-negative $f \in L^p(m) \cap L^2(m)$. Then $u := \overline{P}_t f$ is an element of $\mathcal{D}$. By considering the nest $\{G_n\}$ defined by $G_n := \{\tilde{u} < n\}$ and the associated part space $(\mathcal{Q}^{(n)}, \mathcal{D}^{(n)})$ (as in section 3), we can (and do) assume that u is bounded. Then $u \in L^p(m)$, and both u^{p-1} and $u^{p/2}$ are elements of $\mathcal{D}$, because $|u(y)^r - u(x)^r| \le \|u\|_\infty^{r-1} |u(y) - u(x)|$ and $|u(x)^r| \le \|u\|_\infty^{r-1} |u(x)|$ for $r \ge 1$. We now compute

$$\begin{aligned} \frac{d}{dt} \int_E [e^{-\alpha_0(p)t} \overline{P}_t f]^p\, dm &= p e^{-\alpha_0(p)pt} \int_E \left[u^{p-1} L^{\mathcal{Q}} u - \alpha_0(p) u^p\right] dm \\ &= -p e^{-\alpha_0(p)pt} \left[\mathcal{Q}(u, u^{p-1}) + \alpha_0(p) \|u\|_p^p\right], \end{aligned} \tag{4.1}$$

to see that it suffices to check that

$$\mathcal{Q}(u, u^{p-1}) + \alpha_0(p)\|u\|_p^p \geq 0. \tag{4.2}$$

Noting that $\int_E u^{p-2}\, d\mu_{\langle u\rangle} = 8p^{-2}\mathcal{E}(u^{p/2}, u^{p/2}) = 2(p-1)^{-1}\mathcal{E}(u, u^{p-1})$, we see that

$$\begin{aligned}\mathcal{Q}(u, u^{p-1}) &= 4(p-1)p^{-2}\mathcal{E}(u^{p/2}, u^{p/2}) - \int_E \tilde{u}^{p-1}\, d\mu_{\langle M^{[u]}, M\rangle} \\ &\qquad - \int_E \tilde{u}\, d\mu_{\langle M^{[u^{p-1}]}, \hat{M}\rangle} - \int_E \tilde{u}^p\, d\nu \\ &\geq \left(4(pq)^{-1} - \delta_0(p)\right)\mathcal{E}(u^{p/2}, u^{p/2}) - \alpha_0(p)\|u\|_p^p \\ &\geq -\alpha_0(p)\|u\|_p^p.\end{aligned} \tag{4.3}$$

Thus, for non-negative $f \in L^p(m) \cap L^2(m)$ we have

$$\frac{d}{dt}\int_E |e^{-\alpha_0(p)t}\overline{P}_t f|^p\, dm \leq 0, \quad t \geq 0,$$

which implies that $\|e^{-\alpha_0(p)t}\overline{P}_t f\|_p \leq \|f\|_p$ for such f. By an obvious approximation argument this inequality extends to all $f \in L^p(m)$. Thus each $\overline{P}_t$ maps $L^p(m)$ into itself and (1.16) holds.

For finite p, to prove the strong continuity it is enough, by [51, IX.1], to establish weak continuity; that is, the continuity of $t \mapsto (g, \overline{P}_t f)_m$ for all $f \in L^p(m)$ and $g \in L^q(m)$. In view of the norm estimate proved in the preceding paragraph, we need only consider f and g in $L^1(m) \cap L^\infty(m)$. But $L^1(m) \cap L^\infty(m) \subset L^2(m)$, so the required weak continuity follows from the strong continuity of $(\overline{P}_t)$ on $L^2(m)$ asserted in Theorem 1.1.

5. Examples

Example 5.1 (Sobolev inequality, *cf.* [30, 44])**.** Fix $d > 2$. Assume that a quasi-regular strongly local Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(m)$ satisfies the following Sobolev inequality: There is a constant $C_d \in]0, \infty[$ such that $\mathcal{D} \subset L^{\frac{2d}{d-2}}(m)$ and for $u \in \mathcal{D}$

$$\|u\|^2_{\frac{2d}{d-2}} \leq C_d\, \mathcal{E}(u, u). \tag{5.1}$$

More concretely, let E be a smooth complete d-dimensional Riemannian manifold (M, g) without boundary, with m the volume element of (M, g). It is known that if the Ricci curvature is bounded below and if the injectivity radius of (M, g) is positive, then the following global Sobolev inequality holds (see [20]): $H^1(M) \subset L^{\frac{2d}{d-2}}(M; m)$ and for $u \in H^1(M)$

$$\|u\|^2_{\frac{2d}{d-2}} \leq C_d \int_M |\nabla u|_g^2\, dm, \tag{5.2}$$

which is (5.1) for the Dirichlet form $\mathcal{E}(u, v) := \frac{1}{2}\int_E \langle \nabla u, \nabla v\rangle_g\, dm$ naturally associated with (M, g). The Sobolev inequality (5.1) yields the following estimate (see Theorem 2.4.2 and its proof in [9]): for $\alpha \geq 1$, there exists a constant $C_{\alpha,d} \in]0, \infty[$ such that

$$\|P_t\|_{\alpha \to \infty} \leq C_{\alpha,d}\, t^{-\frac{d}{2\alpha}}, \quad \forall t > 0, \tag{5.3}$$

where $\|P_t\|_{\alpha\to\infty}$ denotes the operator norm of the semigroup $(P_t)_{t>0}$ from $L^\alpha(m)$ to $L^\infty(m)$. Then, for $f \in L^p(m)$ with $p \geq d$, $f^2 m$ is of the Hardy class and $\delta(f^2 m)$ can be taken to be arbitrarily small. Indeed, when $p > d$, f^2m is of the Kato class because

$$
\begin{aligned}
|\langle f^2 m U_\alpha, h\rangle| &\le \int_0^\infty e^{-\alpha t}\|P_t f^2\|_\infty \|h\|_1\,dt \\
&\le \int_0^\infty e^{-\alpha t} t^{-\frac{d}{p}}\|f\|_p^2\|h\|_1\,dt \\
&\le \alpha^{\frac{d}{p}-1}\int_0^\infty e^{-s}s^{-\frac{d}{p}}\,ds\|f\|_p^2\|h\|_1 \quad \text{for } h\in L^1(m).
\end{aligned}
\tag{5.4}
$$

When $p = d$, f^2m is not of the Kato class in general; however, for $u \in \mathcal{D}$,

$$
\begin{aligned}
\int_E u^2 f^2\,dm &= \int_{\{|f|\le N\}} u^2 f^2\,dm + \int_{\{|f|>N\}} u^2 f^2\,dm \\
&\le N^2\|u\|_2^2 + \|I_{\{|f|>N\}}f\|_d^2\|u\|^2_{\frac{2d}{d-2}} \\
&\le N^2\|u\|_2^2 + C_d\|I_{\{|f|>N\}}f\|_d^2\mathcal{E}(u,u),
\end{aligned}
\tag{5.5}
$$

which implies that $\delta(f^2m) := C_d\|I_{\{|f|>N\}}f\|_d^2$ can be made as small as we please by choosing N sufficiently large.

Suppose that $(\mathcal{E},\mathcal{D})$ has the following expression:

$$
\mathcal{E}(u,v) = \tfrac{1}{2}\int_E \Gamma(u,v)\,dm \quad \text{for } u,v\in\mathcal{D};
\tag{5.6}
$$

that is, the energy measure $\mu_{\langle u,v\rangle}$, $u,v\in\mathcal{D}$ has a density $\Gamma(u,v)\in L^1(m)$. Then, for each $M\in\mathcal{M}_{\rm loc}$, the Revuz measure $\mu_{\langle M\rangle}$ associated with the quadratic variation process $\langle M\rangle_t$ has the expression $\Gamma(M)m$ by use of [37, Lemma 2.2]. Consider $M,\hat{M}\in\mathcal{M}_{\rm loc}$ and $C_t := \int_0^t c(X_s)\,ds$ and suppose that $\Gamma(M)^{1/2}\in L^p(m)$, $\Gamma(\hat{M})^{1/2}\in L^q(m)$ and $c\in L^{r/2}(m)$ for $p,q,r\ge d$. Then $\mu_{\langle M\rangle}$, $\mu_{\langle \hat{M}\rangle}$ and $|c|m$ are in the Hardy class and δ_0 can be taken to be arbitrarily small. Note that $C_t^+ = \int_0^t c^+(X_s)\,ds$, where $c^+ := \max\{-c,0\}$ and C^+ is the PCAF associated with the positive part ν^+ of the signed smooth measure $\nu := cm$.

Example 5.2 (Defective log-Sobolev inequality, *cf.* [21, 43])**.** Assume that m is a Borel probability measure on E and that the quasi-regular strongly local Dirichlet form $(\mathcal{E},\mathcal{D})$ on $L^2(m)$ satisfies the defective log-Sobolev inequality: there exist constants $\lambda_1,\lambda_2>0$ such that

$$
\int_X |u(x)|^2\log(|u(x)|/\|u\|_2)\,m(dx) \le \lambda_1\mathcal{E}(u,u)+\lambda_2\|u\|_2^2, \quad \forall u\in\mathcal{D}.
$$

We assume (5.6) and take $M,\hat{M}\in\mathcal{M}_{\rm loc}$ and C to be as in the preceding paragraph. Suppose that there exist $p_1,p_2,q_1,q_2\in[1,\infty[$, $c_1\in]0,\infty[$ such that $e^{c_1p_1\Gamma(M)}, e^{c_1p_2\Gamma(\hat{M})}$, $e^{q_1c^+}, e^{q_2c^-}\in L^1(m)$ and

$$
\sqrt{\frac{2\lambda_1}{c_1p_1}}+\sqrt{\frac{2\lambda_1}{c_1p_2}}+\frac{2\lambda_1}{q_1}<1.
\tag{5.7}
$$

Then $\mu_{\langle M\rangle}$, $\mu_{\langle \hat{M}\rangle}$ and $|c|m$ are of the Hardy class with $\delta_0=\sqrt{\frac{2\lambda_1}{c_1p_1}}+\sqrt{\frac{2\lambda_1}{c_1p_2}}+\frac{2\lambda_1}{q_1}$ by virtue of Young's inequality: $st\le s\log s-s+e^t$, $s\ge 0$, $t\in\mathbb{R}$, and the defective log-Sobolev inequality. Hence we recover the main result of [43] under the milder assumption (5.7), because $\delta_0\le 2\lambda_1(\frac{1}{c_1p_1}+\frac{1}{c_1p_2}+\frac{1}{q_1})+\frac{1}{2}$ and the latter quantity was assumed in [43] to be less than 1.

Example 5.3 (Feynman-Kac formula for CAF locally of zero energy, *cf.* [7])**.** Let $(\mathcal{E},\mathcal{D})$ be a quasi-regular strongly local Dirichlet form on $L^2(m)$ as in section 1. Let $M=\hat{M}=-\frac{1}{2}M^{[\rho]}=-M^{[\rho/2]}$ for some $\rho\in\mathcal{D}_{\rm loc}$ and let $C=0$. Suppose that $\mu_{\langle\rho\rangle}\in S_H$ with $\delta(\mu_{\langle\rho\rangle})<1/2$. Then assumption (1.2) is satisfied: $\delta_0=2\sqrt{2\delta(-M^{[\rho/2]})}=2\sqrt{2^{-1}\delta(\mu_{\langle\rho\rangle})}<2\sqrt{4^{-1}}=1$. Since $N^{[\rho]}$ is the even part of $-M^{[\rho]}$, we have $Z_t=\exp[N_t^{[\rho]}]$, $\mathbf{P}^m$-a.s. on $\{t<\zeta\}$, hence $Q_tf(x)=\mathbf{E}^x[\exp(N_t^{[\rho]})f(X_t)]$, m-a.e. $x\in E$ for $f\in L^2(m)$. When $\rho\in\mathcal{D}_e$, $(\mathcal{Q},\mathcal{D})$ has the following expression:

$$\mathcal{Q}(u,v)=\mathcal{E}(u,v)+\mathcal{E}(uv,\rho)\quad\text{ for } u,v\in b\mathcal{D}. \tag{5.8}$$

Here $\mathcal{D}_e$ is the extended Dirichlet space of $(\mathcal{E},\mathcal{D})$; see p. 36 in [17].

Example 5.4 (Hardy inequality in upper half plane)**.** Let $\mathbb{R}^d_+:=\{x=(\bar{x},x_d)\in\mathbb{R}^d\mid\bar{x}\in\mathbb{R}^{d-1},\ x_d>0\}$ be the upper half plane in $\mathbb{R}^d$ $(d\geq 2)$. We consider the classical Dirichlet form $(\mathcal{E},\mathcal{D}):=(\frac{1}{2}\mathbf{D},H^1_0(\mathbb{R}^d_+))$ on $L^2(\mathbb{R}^d_+)$ associated with the Brownian motion $(\Omega,B_t^{\mathbb{R}^d_+},\mathbf{P}^x)$ on $\mathbb{R}^d_+$ with absorbing boundary $\mathbb{R}^d_0:=\{x=(\bar{x},x_d)\in\mathbb{R}^d\mid\bar{x}\in\mathbb{R}^{d-1},\ x_d=0\}$.

Then the following Hardy inequality holds (see [36, Section 2.1.6], [10, Thm. 6]):

$$\int_{\mathbb{R}^d_+}\frac{u(x)^2}{|x_d|^2}\,dx\leq 4\int_{\mathbb{R}^d_+}|\nabla u(x)|^2\,dx\quad\text{for all } u\in H^1_0(\mathbb{R}^d_+). \tag{5.9}$$

Fix $\varepsilon>0$ and let $b(x):=(4\sqrt{2}(2+\varepsilon)x_d)^{-1}\mathbf{e}_1$, $\hat{b}(x):=-(4\sqrt{2}(2+\varepsilon)x_d)^{-1}\mathbf{e}_d$, and $c(x):=(2(4+\varepsilon))^{-1}x_d^{-2}$, where $\mathbf{e}_1:=(1,0,\cdots,0)$, $\mathbf{e}_d:=(0,\cdots,0,1)$. We consider $M_t:=\int_0^{t\wedge\tau^+}b(B_s)\,dB_s$, $\hat{M}_t:=\int_0^{t\wedge\tau^+}\hat{b}(B_s)\,dB_s$ and $C_t:=\int_0^{t\wedge\tau^+}c(B_s)\,ds$, where $\tau^+:=\tau(\mathbb{R}^d_+)$ is the first exit time of d-dimensional Brownian motion $(\Omega,B_t,\mathbf{P}^x)$ from $\mathbb{R}^d_+$. Then our assumptions are satisfied with $\delta_0=1/(2+\varepsilon)+2/(4+\varepsilon)<1$. Note $|b|,|\hat{b}|,c\notin L^p(\mathbb{R}^d_+)$ for any $p>0$.

Example 5.5 (*cf.* [18],[24],[6], [40], [35], (4.2) in [14])**.** When $d\geq 3$, the following Hardy inequality holds: for any $r>0$

$$\int_{B_r(0)}\frac{u(x)^2}{|x|^2}\,dx\leq\frac{4}{(d-2)^2}\int_{B_r(0)}|\nabla u(x)|^2\,dx\quad\text{for } u\in H^1_0(B_r(0)). \tag{5.10}$$

Hence

$$\int_{\mathbb{R}^d}\frac{u(x)^2}{|x|^2}\,dx\leq\frac{4}{(d-2)^2}\int_{\mathbb{R}^d}|\nabla u(x)|^2\,dx\quad\text{for } u\in H^1(\mathbb{R}^d). \tag{5.11}$$

Take $\varphi\in C^\infty([0,\infty[)$ with $0<\varphi\leq 1$, $\varphi\equiv 1$ on $[0,1/2]$, and $\varphi(s)\leq 1/s^d$ for $s\in[1,\infty[$. Let $b(x)=\hat{b}(x):=\frac{d-2}{8\sqrt{2}+\varepsilon}\frac{\varphi(|x|)}{|x|^2}x$ and $c(x):=\left(\frac{d-2}{2\sqrt{2}+\varepsilon}\right)^2\frac{\varphi(|x|)^2}{|x|^2}$ for $\varepsilon>0$. We consider $M_t:=\int_0^t b(B_s)\,dB_s$, $\hat{M}_t:=\int_0^t\hat{b}(B_s)\,dB_s$ and $C_t:=\int_0^t c(B_s)\,ds$. Then $|b|^2dx$, $|\hat{b}|^2dx$ and $|c|dx$ are of the Hardy class with respect to the Dirichlet form associated with d-dimensional Brownian motion $\mathbf{X}:=(\Omega,B_t,\mathbf{P}^x)$ and $\delta_0=4\sqrt{2}/(8\sqrt{2}+\varepsilon)+4/(2\sqrt{2}+\varepsilon)^2<1$. Note that $|b|,|\hat{b}|,|c|^{1/2}\notin L^p(\mathbb{R}^d)$ for $p\geq d$, but $|b|,|\hat{b}|,|c|^{1/2}\in L^p(\mathbb{R}^d)$ for $d/(d+1)<p<d$ and $|b|^2,|\hat{b}|^2,|c|$ are not of the Kato class when the support of φ is bounded, by Theorem 1.3(viii) in [1].

Example 5.6 (Low dimensional case)**.** Fix $\alpha \geq 1$ and $r \geq 1$ with $r^{\alpha+1} \geq \alpha$. Let $\psi(x) := 1/|x|^\alpha$, $x \in \mathbb{R}^d$. Then $|\nabla\psi| \leq 1$ and $\frac{1}{2}\Delta\psi \leq \alpha(\alpha+2-d)/2\psi$ on $G := \{y \in \mathbb{R}^d \mid |y| > r\}$. We assume $d = 1$ if $\alpha = 1$, or $d = 1, 2$ if $\alpha > 1$. Then $\alpha(\alpha+2-d)/2 > 1/2$. By (4.2) in [14] we have the following Hardy inequality:

$$\int_G u(x)^2|x|^{2\alpha}\,dx \leq \frac{4}{\alpha^2(\alpha+1-d)^2}\int_G |\nabla u(x)|^2\,dx \quad \text{for } u \in H_0^1(G). \tag{5.12}$$

On the other hand, the smooth measure $|x|^{2\alpha}dx$ is not of the Kato class with respect to the Dirichlet form corresponding to the Brownian motion $(\Omega, B_t^G, \mathbf{P}^x)$ on G with absorbing boundary $\partial B_r(0)$. Indeed, consider

$$\sup_{x\in G}\int_{\{|x-y|<\frac{r}{2}, r\leq d(x,G^c)\wedge d(y,G^c)\}} |y|^{2\alpha}\,dy = \infty, \tag{5.13}$$

where $d(x, G^c) := \inf_{z\in G^c}|x-z|$. Let $p_t^G(x,y)$ on $(0,\infty)\times G\times G$ be the heat kernel for (B_t^G), defined by

$$p_t^G(x,y) := p_t(x,y) - \mathbf{E}^x[p_{t-\tau(G)}(B_{\tau(G)}, y)I_{\{\tau(G)<t\}}], \tag{5.14}$$

where $\mathbf{X} = (\Omega, B_t, \mathbf{P}^x)$ is the d-dimensional Brownian motion, $p_t(x,y) := \frac{1}{(2\pi t)^{d/2}}\exp(-\frac{|x-y|^2}{2t})$, $x, y \in \mathbb{R}^d$, is the heat kernel of $\mathbf{X}$, and $\tau(G)$ is the first exit time of $\mathbf{X}$ from G. Then by [8, Proof of Thm. 2.4], for $0 < s < 1/d$, $|x-y| < r \leq d(x,G^c)\wedge d(y,G^c)$,

$$p_s^G(x,y) \geq \frac{1}{(2\pi s)^{d/2}}(e^{-\frac{|x-y|^2}{2s}} - e^{-\frac{r^2}{2s}}). \tag{5.15}$$

Hence, for $t/2 < s < t < 1/d$, $|x-y| < r/2$ and $r \leq d(x,G^c)\wedge d(y,G^c)$,

$$p_s^G(x,y) \geq \frac{1}{(4\pi t)^{d/2}}(e^{-\frac{r^2}{4t}} - e^{-\frac{r^2}{2t}}) =: C_t > 0. \tag{5.16}$$

Therefore, for $t < 1/d$

$$\begin{aligned}
&\frac{t}{2C_t}\sup_{x\in G}\int_{\{|x-y|<\frac{r}{2}, r\leq d(x,G^c)\wedge d(y,G^c)\}} |y|^{2\alpha}\,dy \\
&\leq \sup_{x\in G}\int_{\{|x-y|<\frac{r}{2}, r\leq d(x,G^c)\wedge d(y,G^c)\}} \left(\int_{t/2}^t p_s^G(x,y)ds\right)|y|^{2\alpha}\,dy \\
&\leq \sup_{x\in G}\mathbf{E}^x\left[\int_0^t |B_s^G|^{2\alpha}\,ds\right],
\end{aligned} \tag{5.17}$$

which implies that $|x|^{2\alpha}dx$ is not of the Kato class (or even of the Dynkin class). However, if $b(x) := \hat{b}(x) := \alpha(\alpha+1-d)x|x|^{\alpha-1}/(8\sqrt{2}+\varepsilon)$ and $c(x) := \alpha^2(\alpha+1-d)^2|x|^{2\alpha}/(8+\varepsilon)$, then our assumptions are satisfied with $\delta_0 = 4\sqrt{2}/(8\sqrt{2}+\varepsilon) + 4/(8+\varepsilon) < 1$.

References

[1] M. Aizenman and B. Simon, Brownian motion and Harnack inequality for Schrödinger operators. *Comm. Pure Appl. Math.* **35** (1982), 209–273.

[2] S. Albeverio and Z.M. Ma, Perturbation of Dirichlet forms—lower semiboundedness, closability, and form cores, *J. Funct. Anal.* **99** (1991), 332–356.

[3] S. Albeverio and Z.M. Ma, A general correspondence between Dirichlet forms and right processes, *Bull. Amer. Math. Soc.* **26** (1992), 245–252.

[4] S. Albeverio, Z.M. Ma and M. Röckner, Quasi-regular Dirichlet forms and Markov processes, *J. Funct. Anal.* **111** (1993), 118–154.

[5] A.G. Belyĭ and Yu.A. Semenov, On the L^p-theory of Schrödinger semigroups. II, *Sibirsk. Mat. Zh.* **31** (1990), 16–26; translation in *Siberian Math. J.* **31** (1990), 540–549.

[6] G. Carron, Inégalités de Hardy sur les variétés riemanniennes non-compactes. (French) [Hardy inequalities on noncompact Riemannian manifolds] *J. Math. Pures Appl. (9)* **76** (1997), 883–891.

[7] Z.Q. Chen and T.S. Zhang, Girsanov and Feynman-Kac type transformations for symmetric Markov processes, *Ann. Inst. H. Poincaré Probab. Statist.* **38** (2002), 475–505.

[8] K.L. Chung and Z.X. Zhao, *From Brownian motion to Schrödinger's equation*. Springer-Verlag, Berlin, 1995.

[9] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1990.

[10] E.B. Davies, A review of Hardy inequalities, *The Maz'ya Anniversary Collection, Vol. 2 (Rostock, 1998)*, pp. 55–67, Birkhäuser, Basel, 1999.

[11] C. Dellacherie and P.-A. Meyer, *Probabilités et Potentiel, Chapitres V á VIII*. Hermann, Paris, 1980.

[12] P.J. Fitzsimmons, Even and odd continuous additive functionals. *Dirichlet forms and stochastic processes* (Beijing, 1993), 139–154, de Gruyter, Berlin, 1995.

[13] P.J. Fitzsimmons, Absolute continuity of symmetric diffusions, *Ann. Prob.* **25** (1997), 230–258.

[14] P.J. Fitzsimmons, Hardy's inequality for Dirichlet forms, *J. Math. Anal. Appl.* **250** (2000), 548–560.

[15] P.J. Fitzsimmons, On the quasi-regularity of semi-Dirichlet forms, *Potential Anal.* **15** (2001), 151–185.

[16] P.J. Fitzsimmons and R.K. Getoor, On the potential theory of symmetric Markov processes, *Math. Ann.* **281** (1988), 495–512. set function,

[17] M. Fukushima, Y. Ōshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter, Berlin, 1994.

[18] N. Garofalo and F.H. Lin, Unique continuation for elliptic operators: a geometric-variational approach, *Comm. Pure Appl. Math.* **40** (1987), 347–366.

[19] R.K. Getoor, Measure perturbations of Markovian semigroups, *Potential Anal.* **11** (1999), 101–133.

[20] E. Hebey, Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius, *Amer. J. Math.* **118** (1996), 291–300.

[21] M. Hino, Existence of invariant measures for diffusion processes on a Wiener space, *Osaka J. Math.* **35** (1998), 717–734.

[22] T. Kato, *Perturbation Theory for Linear Operators* (Second edition, corrected printing). Springer-Verlag, Berlin-New York. 1980.

[23] V.F. Kovalenko and Yu.A. Semenov, C_o-semigroups in the spaces $L^p(\mathbf{R}^d)$ and $\widehat{C}(\mathbf{R}^d)$ generated by the differential expression $\Delta + b \cdot \nabla$, *Teor. Veroyatnost. i Primenen.* **35** (1990), 449–458; translation in *Theory Probab. Appl.* **35** (1990), 443–453.

[24] J.L. Kazdan, Unique continuation in geometry, *Comm. Pure Appl. Math.* **41** (1988), 667–681.

[25] K. Kuwae, Maximum principles for subharmonic functions via local semi-Dirichlet forms, preprint, 2002.

[26] K. Kuwae, Conservativeness of diffusion processes by a drift transformation, preprint, 2002.

[27] V. Liskevich, On C_0-semigroups generated by elliptic second order differential expressions on L^p-spaces, *Differential Integral Equations* **9** (1996), 811–826.

[28] V.A. Liskevich, M.A. Perelmuter, and Yu.A. Semenov, Form-bounded perturbations of generators of sub-Markovian semigroups, *Acta Appl. Math.* **44** (1996), 353–377.

[29] V. Liskevich, Z. Sobol, and H. Vogt, On the L_p-theory of C_0-semigroups associated with second oreder elliptic operators. II, *J. Funct. Anal.* **193** (2002), 55–76.

[30] J. Lunt, T.J. Lyons and T.S. Zhang, Integrability of functionals of Dirichlet processes, probabilistic representations of semigroups, and estimates of heat kernels, *J. Funct. Anal.* **153** (1998), 320–342.

[31] T.J. Lyons and T.S. Zhang, Decomposition of Dirichlet processes and its application, *Ann. Prob.* **22** (1994), 494–524.

[32] T.J. Lyons and T.S. Zhang, Convergence of non-symmetric Dirichlet processes, *Stochastics Stochastics Rep.* **57** (1996), 159–167.

[33] Z.M. Ma and M. Röckner, *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Springer-Verlag, Berlin, 1992.

[34] Z.M. Ma and M. Röckner, Markov processes associated with positivity preserving coercive forms, *Canad. J. Math.* **47** (1995), 817–840.

[35] M. Marcus, V.J. Mizel, and Y. Pinchover, On the best constant for Hardy's inequality in $\mathbb{R}^n$, *Trans. Amer. Math. Soc.* **350** (1998), 3237–3255.
[36] V.G. Maz'ya, *Sobolev spaces*, Springer-Verlag, Berlin, 1985.
[37] S. Nakao, Stochastic calculus for continuous additive functionals of zero energy, *Z. Wahrsch. verw. Gebiete* **68** (1985), 557–578.
[38] S. Nakao, Girsanov formula in Dirichlet space theory, *The Science Reports of Kanazawa University*, **46** (2001), 1–7.
[39] S. Nakao, Girsanov formula in Dirichlet space theory, preprint, 2002.
[40] B. Opic and A. Kufner, *Hardy-type Inequalities*, Longman, Essex, 1990.
[41] Y. Ōshima and T. Yamada, On some representations of continuous additive functionals locally of zero energy, *J. Math. Soc. Japan* **36** (1984), 315–339.
[42] M. Röckner and T.S. Zhang, Convergence of operators semigroups generated by elliptic operators, *Osaka J. Math.* **34** (1997), 923–932.
[43] M. Röckner and T.S. Zhang, Probabilistic representations and hyperbound estimates for semigroups, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2** (1999), 337–358.
[44] M. Röckner and T.S. Zhang, Lower order perturbations of Dirichlet processes, 2001, to appear in *Forum Math.*
[45] M. Sharpe, *General Theory of Markov Processes*, Academic Press, San Diego, 1988.
[46] I. Shigekawa, A non-symmetric diffusion process on the Wiener space, preprint, 1998.
[47] B. Simon, *Functional Integration and Quantum Physics*, Academic Press, New York-London, 1979.
[48] Z. Sobol and H. Vogt, On the L_p-theory of C_0-semigroups associated with second oreder elliptic operators. I, *J. Funct. Anal.* **193** (2002), 24–54.
[49] P. Stollmann and J. Voigt, Perturbation of Dirichlet forms by measures, *Potential Anal.* **5** (1996), 109–138.
[50] J.B. Walsh, Markov processes and their functionals in duality, *Z. Wahrsch. verw. Gebiete* **24** (1972), 229–246.
[51] K. Yosida, *Functional Analysis* (Sixth edition), Springer-Verlag, Berlin-New York, 1980.
[52] T.S. Zhang, Generalized Feynman-Kac semigroups, associated quadratic forms and asymptotic properties, *Potential Anal.* **14** (2001), 387–408.

P.J. Fitzsimmons — Department of Mathematics, University of California, San Diego, 9500 Gilman Drive, La Jolla, California 92093-0112, USA
E-mail address: pfitz@math.ucsd.edu

K. Kuwae — Department of Mathematical Sciences, Yokohama City University, Yokohama 236-0027, JAPAN
E-mail address: kuwae@yokohama-cu.ac.jp