On a result of A. Jakubowski

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Chebyshev's other inequality is the statement that if f and g are non-decreasing functions from $\mathbf{R}$ to $\mathbf{R}$ and if X is a random variable such that both $f(X)$ and $g(X)$ are square integrable, then $\mathrm{Cov}[f(X), g(X)] \geq 0$. Jakubowski [**J**] has recently discussed the equality case, showing that if $\mathrm{Cov}[f(X), g(X)] = 0$ then either $f(X)$ is constant a.s. or $g(X)$ is constant a.s. I record here another proof of this assertion.

The neatest proof of the inequality goes as follows. Let Y be an independent copy of X, and notice that

$$[f(X) - f(Y)] \cdot [g(X) - g(Y)] \geq 0 \tag{1}$$

everywhere, and so

$$\mathrm{Cov}[f(X), g(X)] = \frac{1}{2}\mathbf{E}\left[[f(X) - f(Y)] \cdot [g(X) - g(Y)]\right] \geq 0. \tag{2}$$

As noted in [**J**], this approach also gives insight into the case in which $f(X)$ and $g(X)$ are uncorrelated.

First notice that $f(X)$ is constant a.s. if and only if $\mathrm{Var}[f(X)] = 0$. By $f = g$ case of (2) this in turn is equivalent to

$$\mathbf{E}\left[[f(X) - f(Y)]^2\right] = 0. \tag{3}$$

Thus, assuming $\mathrm{Cov}[f(X), g(X)] = 0$ we deduce from (1) and (2) that

$$[f(X) - f(Y)] \cdot [g(X) - g(Y)] = 0, \quad \text{a.s.} \tag{4}$$

We now argue that if $\mathrm{Var}[f(X)] > 0$ then $\mathrm{Var}[g(X)] = 0$.

Suppose that $f(X)$ and $g(X)$ are uncorrelated and that $\mathrm{Var}[f(X)] > 0$. Then $\mathbf{E}\left[[f(X) - f(Y)]^2\right] > 0$, so $\mathbf{P}[f(X) < f(Y)] = \frac{1}{2}\mathbf{P}[f(X) \neq f(Y)] > 0$. Let $A_0 := \{f(X) < f(Y)\}$. In view of (4) this differs from $A := \{f(X) < f(Y), g(X) = g(Y)\}$ by a null set.

Now introduce a third random variable Z independent of (X, Y) with the same distribution as X and Y. Consider the event

$$B := A \cap \{g(Y) < g(Z)\}. \tag{5}$$

On B we have $g(X) < g(Z)$; by (4) stated for the pair (X, Z) we have $f(X) = f(Z)$ a.s. on B. Then, by the monotonicity of f, $f(X) = f(Y)$ a.s. on B. It follows that $\mathbf{P}[B] = 0$. Similarly, $\mathbf{P}[A \cap \{g(Z) < g(X)\}] = 0$. Thus,

$$g(X) = g(Y) = g(Z), \quad \text{a.s. on } A. \tag{6}$$

Let μ denote the common distribution of X, Y, Z. From (6) and the fact that Z is independent of A, we deduce that

$$g(X) = g(Y) = g(z), \quad \text{a.s. on } A, \tag{7}$$

for μ-a.e. $z \in \mathbf{R}$. Let J denote the smallest interval carrying all the mass of μ. Then because g is monotone we deduce from (7) that

$$g(X) = g(Y) = g(z), \forall z \in J, \quad \text{a.s. on } A. \tag{8}$$

But $\mathbf{P}[A] > 0$ so A has at least one element; consequently, (8) implies that g is constant on J. Consequently, $g(X)$ is constant a.s. $\square$

References

[J] Jakubowski, A.: A complement to the Chebyshev integral inequality, *Stat. Probab. Letters* **168** (2021) 108934.