FORTY-FIRST ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 6, 1980 Examination A

A1. Let b and c be fixed real numbers and let the ten points (j, y_j), $j = 1, 2, \dots, 10$, lie on the parabola $y = x^2 + bx + c$. For $j = 1, 2, \dots, 9$, let I_j be the point of intersection of the tangents to the given parabola at (j, y_j) and $(j+1, y_{j+1})$. Determine the polynomial function $y = g(x)$ of least degree whose graph passes through all nine points I_j.

A2. Let r and s be positive integers. Derive a formula for the number of ordered quadruples (a, b, c, d) of positive integers such that

$$3^r \cdot 7^s = \mathrm{lcm}[a, b, c] = \mathrm{lcm}[a, b, d] = \mathrm{lcm}[a, c, d] = \mathrm{lcm}[b, c, d].$$

The answer should be a function of r and s.
(Note that $\mathrm{lcm}[x, y, z]$ denotes the least common multiple of x, y, z.)

A3. Evaluate

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

A4.
(a) Prove that there exist integers a, b, c, not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$

(b) Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that

$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}.$$

A5. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t) \sin t \, dt = \int_0^x P(t) \cos t \, dt$$

has only finitely many real solutions x.

A6. Let C be the class of all real valued continuously differentiable functions f on the interval $0 \le x \le 1$ with $f(0) = 0$ and $f(1) = 1$. Determine the largest real number u such that

$$u \le \int_0^1 |f'(x) - f(x)| \, dx$$

for all f in C.

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Saturday, December 6, 1980 Examination B

B1. For which real numbers c is $(e^x + e^{-x})/2 \leq e^{cx^2}$ for all real x.

B2. Let S be the solid in three-dimensional space consisting of all points (x, y, z) satisfying the following system of six simultaneous conditions:

$$x \geq 0, \quad y \geq 0, \quad z \geq 0,$$
$$x + y + z \leq 11,$$
$$2x + 4y + 3z \leq 36,$$
$$2x + 3z \leq 24.$$

(a) Determine the number of vertices v of S.
(b) Determine the number of edges e of S.
(c) Sketch in the bc-plane the set of points (b, c) such that $(2, 5, 4)$ is one of the points (x, y, z) at which the linear function $bx + cy + z$ assumes its maximum value on S.

B3. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$?
(Express the answer in the simplest form.)

B4. Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leq i \leq 1066$. Prove that there exist ten elements $x_1, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, \ldots, x_{10}$.
(Here $|S|$ means the number of elements in the set S.)

B5. For each $t \geq 0$, let S_t be the set of all nonnegative, increasing, convex, continuous, real-valued functions $f(x)$ defined on the closed interval $[0, 1]$ for which

$$f(1) - 2f(2/3) + f(1/3) \geq t[f(2/3) - 2f(1/3) + f(0)].$$

Develop necessary and sufficient conditions on t for S_t to be closed under multiplication. (This closure means that, if the functions $f(x)$ and $g(x)$ are in S_t, so is their product $f(x)g(x)$. A function $f(x)$ is convex if and only if $f(su + (1-s)v) \leq sf(u) + (1-s)f(v)$ whenever $0 \leq s \leq 1$.)

B6. An infinite array of rational numbers $G(d, n)$ is defined for integers d and n with $1 \leq d \leq n$ as follows:

$$G(1, n) = \frac{1}{n}, \quad G(d, n) = \frac{d}{n} \sum_{i=d}^{n} G(d-1, i-1) \text{ for } d > 1.$$

For $1 < d \leq p$ and p *prime*, prove that $G(d, p)$ is expressible as a quotient s/t of integers s and t with t *not* an integral multiple of p.
(For example, $G(3, 5) = 7/4$ with the denominator 4 not a multiple of 5.)