A1. Let $b$ and $c$ be fixed real numbers and let the ten points $(j, y_j)$, $j = 1, 2, \ldots, 10$, lie on the parabola $y = x^2 + bx + c$. For $j = 1, 2, \ldots, 9$, let $I_j$ be the point of intersection of the tangents to the given parabola at $(j, y_j)$ and $(j + 1, y_{j+1})$. Determine the polynomial function $y = g(x)$ of least degree whose graph passes through all nine points $I_j$.

A2. Let $r$ and $s$ be positive integers. Derive a formula for the number of ordered quadruples $(a, b, c, d)$ of positive integers such that

$$3^r \cdot 7^s = \text{lcm}[a, b, c] = \text{lcm}[a, b, d] = \text{lcm}[a, c, d] = \text{lcm}[b, c, d].$$

The answer should be a function of $r$ and $s$.
(Note that lcm$[x, y, z]$ denotes the least common multiple of $x, y, z$.)

A3. Evaluate

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^2}.$$ 

A4. (a) Prove that there exist integers $a, b, c$, not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$ 

(b) Let $a, b, c$ be integers, not all zero and each of absolute value less than one million. Prove that

$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}.$$ 

A5. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t) \sin t \, dt = \int_0^x P(t) \cos t \, dt$$

has only finitely many real solutions $x$.

A6. Let $C$ be the class of all real valued continuously differentiable functions $f$ on the interval $0 \leq x \leq 1$ with $f(0) = 0$ and $f(1) = 1$. Determine the largest real number $u$ such that

$$u \leq \int_0^1 |f'(x) - f(x)| \, dx$$

for all $f$ in $C$. 

B1. For which real numbers $c$ is $(e^x + e^{-x})/2 \leq e^{cx^2}$ for all real $x$.

B2. Let $S$ be the solid in three-dimensional space consisting of all points $(x, y, z)$ satisfying the following system of six simultaneous conditions:

\begin{align*}
    &x \geq 0, \quad y \geq 0, \quad z \geq 0, \\
    &x + y + z \leq 11, \\
    &2x + 4y + 3z \leq 36, \\
    &2x + 3z \leq 24.
\end{align*}

(a) Determine the number of vertices $v$ of $S$.
(b) Determine the number of edges $e$ of $S$.
(c) Sketch in the $bc$-plane the set of points $(b, c)$ such that $(2, 5, 4)$ is one of the points $(x, y, z)$ at which the linear function $bx + cy + z$ assumes its maximum value on $S$.

B3. For which real numbers $a$ does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$?
(Express the answer in the simplest form.)

B4. Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set $X$ such that $|A_i| > \frac{1}{2}|X|$ for $1 \leq i \leq 1066$. Prove that there exist ten elements $x_1, \ldots, x_{10}$ of $X$ such that every $A_i$ contains at least one of $x_1, \ldots, x_{10}$.
(Here $|S|$ means the number of elements in the set $S$.)

B5. For each $t \geq 0$, let $S_t$ be the set of all nonnegative, increasing, convex, continuous, real-valued functions $f(x)$ defined on the closed interval $[0, 1]$ for which

\[ f(1) - 2f(2/3) + f(1/3) \geq t[f(2/3) - 2f(1/3) + f(0)]. \]

Develop necessary and sufficient conditions on $t$ for $S_t$ to be closed under multiplication.
(This closure means that, if the functions $f(x)$ and $g(x)$ are in $S_t$, so is their product $f(x)g(x)$. A function $f(x)$ is convex if and only if $f(su + (1-s)v) \leq sf(u) + (1-s)f(v)$ whenever $0 \leq s \leq 1$.)

B6. An infinite array of rational numbers $G(d, n)$ is defined for integers $d$ and $n$ with $1 \leq d \leq n$ as follows:

\[ G(1, n) = \frac{1}{n}, \quad G(d, n) = \frac{d}{n} \sum_{i=d}^{n} G(d-1, i-1) \text{ for } d > 1. \]

For $1 < d \leq p$ and $p$ prime, prove that $G(d, p)$ is expressible as a quotient $s/t$ of integers $s$ and $t$ with $t$ not an integral multiple of $p$.
(For example, $G(3, 5) = 7/4$ with the denominator 4 not a multiple of 5.)