# FORTY-THIRD ANNUAL
# WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 4, 1982 Examination A

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**A-1.** Let $V$ be the region in the cartesian plane consisting of all points $(x, y)$ satisfying the simultaneous equations

$$|x| \leq y \leq |x| + 3 \quad \text{and} \quad y \leq 4.$$

Find the centroid $(\overline{x}, \overline{y})$ of $V$.

**A-2.** For positive real $x$. let

$$B_n(x) = 1^x + 2^x + 3^x + \cdots + n^x.$$

Prove or disprove the convergence of

$$\sum_{n=2}^{\infty} \frac{B_n(\log_n 2)}{(n \log_2 n)^2}.$$

**A-3.** Evaluate

$$\int_0^{\infty} \frac{\operatorname{Arctan}(\pi x) - \operatorname{Arctan}(x)}{x}\, dx.$$

**A-4.** Assume that the system of simultaneous differential equations

$$y' = -z^3, \quad z' = y^3$$

with the initial conditions $y(0) = 1$, $z(0) = 0$ has a unique solution $y = f(x)$, $z = g(x)$ defined for all real $x$. Prove that there exists a positive constant $L$ such that for all real $x$,

$$f(x + L) = f(x), \quad g(x + L) = g(x).$$

**A-5.** Let $a$, $b$, $c$, and $d$ be positive integers and

$$r = 1 - \frac{a}{b} - \frac{c}{d}.$$

Given that $a + c \leq 1982$ and $r > 0$, prove that

$$r > \frac{1}{1983^3}.$$

**A-6.** Let $\sigma$ be a bijection of the positive integers, that is, a one-to-one function from $\{1, 2, 3, \ldots\}$ onto itself. Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers with the following properties:

(i) $|x_n|$ is a strictly decreasing function of $n$;

(ii) $|\sigma(n) - n| \cdot |x_n| \to 0$ as $n \to \infty$;

(iii) $\lim_{n\to\infty} \sum_{k=1}^{n} x_k = 1$.

Prove or disprove that these conditions imply that

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = 1.$$

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**B-1.** Let $M$ be the midpoint of side $BC$ of a general $\triangle ABC$. Using the *smallest possible* $n$, describe a method for cutting $\triangle AMB$ into $n$ triangles which can be reassembled to form a triangle congruent to $\triangle AMC$.

**B-2.** Let $A(x,y)$ denote the number of points $(m,n)$ in the plane with integer coordinates $m$ and $n$ satisfying $m^2+n^2 \le x^2+y^2$. Let $g=\sum_{k=0}^{\infty} e^{-k^2}$. Express

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x,y)e^{-x^2-y^2}\,dx\,dy$$

as a polynomial in $g$.

**B-3.** Let $p_n$ be the probability that $c+d$ is a perfect square when the integers $c$ and $d$ are selected independently at random from the set $\{1,2,3,\ldots,n\}$. Show that $\lim_{n\to\infty}(p_n\sqrt{n})$ exists and express this limit in the form $r(\sqrt{s}-t)$, where $s$ and $t$ are integers and $r$ is a rational number.

**B-4.** Let $n_1, n_2, \ldots n_s$ be distinct integers such that

$$(n_1+k)(n_2+k)\cdots(n_s+k)$$

is an integral multiple of $n_1 n_2 \cdots n_s$ for every integer $k$. For each of the following assertions, give a proof or a counterexample:

(a) $|n_i|=1$ for some $i$.

(b) If further all $n_i$ are positive, then

$$(n_1,n_2,\ldots,n_s)=(1,2,\ldots,s).$$

**B-5.** For each $x>e^e$ define a sequence $S_x=u_0,u_1,u_2,\ldots$ recursively as follows: $u_0=e$, while for $n\ge 0$, $n+n+1$ is the logarithm of $x$ to the base $u_n$. Prove that $S_x$ converges to a number $g(x)$ and that the function $g$ defined in this way is continuous for $x>e^e$.

**B-6.** Let $K(x,y,z)$ denote the area of a triangle whose sides have lengths $x$, $y$, and $z$. For any two triangles with sides $a$, $b$, $c$ and $a'$, $b'$, $c'$, respectively, prove that

$$\sqrt{K(a,b,c)}+\sqrt{K(a',b',c')}|le\sqrt{K(a+a',b+b',c+c')}$$

and determine the cases of equality.