FORTY-FOURTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 3, 1983 — Examination A

A-1. How many positive integers n are there such that n is an exact divisor of at least one of the numbers

$$10^{40}, 20^{30} \,?$$

A-2. The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

A-3. Let p be in the set $\{3, 5, 7, 11, \ldots\}$ of odd primes and let

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0, 1, 2, \ldots, p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p, that is, $F(a) - F(b)$ is not exactly divisible by p.

A-4. Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero. [As usual, $\binom{m}{r} = \frac{m!}{r!(m-r)!}$.]

A-5. Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integers n.

Here $[x]$ denotes the greatest integer less than or equal to x.

A-6. Let $\exp(t)$ denote e^t and

$$F(x) = \frac{x^4}{\exp(x^3)} \int_0^x \int_0^{x-u} \exp(u^3 + v^3)\, dv\, du.$$

Find $\lim_{x\to\infty} F(x)$ or prove that it does not exist.

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Saturday, December 3, 1983 — Examination B

B-1. Let v be a vertex (corner) of a cube C with edges of length 4. Let S be the largest sphere that can be inscribed in C. Let R be the region consisting of all points p between S and C such that p is closer to v that to any other vertex of the cube. Find the volume of R.

B-2. For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than three times. For example $C(8) = 5$ since the representations for 8 are:

$$8, \quad 4+4, \quad 4+2+2, \quad 4+2+1+1, \quad 2+2+2+1+1.$$

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = [P(n)]$ for all positive integers n; here $[u]$ denotes the greatest integer less than or equal to u.

B-3. Assume that the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

has solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ on the whole real line such that

$$y_1^2(x) + y_2^2(x) + y_3^2(x) = 1$$

for all real x. Let

$$f(x) = (y_1'(x))^2 + (y_2'(x))^2 + (y_3'(x))^2 .$$

Find constants A and B such that $f(x)$ is a solution to the differential equation

$$y'(x) + Ap(x)y = Br(x).$$

B-4. Let $f(n) = n + [\sqrt{n}]$ where $[x]$ is the largest integer less than or equal to x. Prove that, for every positive integer m, the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \dots$$

contains at least one square of an integer.

B-5. Let $\|u\|$ denote the distance from the real number u to the nearest integer. (For example, $\|2.8\| = .2 = \|3.2\|$.) For positive integers n, let

$$a_n = \frac{1}{n}\int_1^n \left\|\frac{n}{x}\right\| dx.$$

Determine $\lim_{n\to\infty} a_n$. You may assume the identity

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots = \frac{\pi}{2}.$$

B-6. Let k be a positive integer, let $m = 2^k + 1$, and let $r \neq 1$ be a complex root of $z^m - 1 = 0$. Prove that there exists polynomials $P(z)$ and $Q(z)$ with integer coefficients such that

$$(P(r))^2 + (Q(r))^2 = -1.$$