FORTY-SIXTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

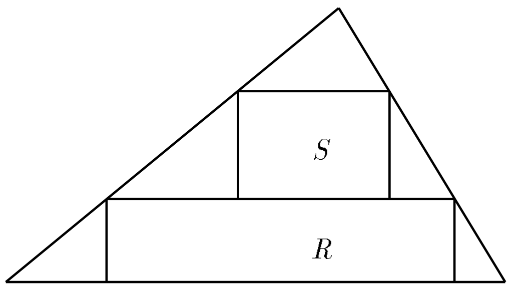
Saturday, December 7, 1985 Examination A

A-1. Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
and
(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$,

where $\emptyset$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c and d are nonnegative integers.

A-2. Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown:



Let $A(X)$ denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\dfrac{A(R) + A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

A-3. Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

A-4. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

A-5. Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1 \leq m \leq 10$, is $I_m \neq 0$?

A-6. If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial with real coefficients a_i, then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $f(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

(i) $g(0) = 1$,
and
(ii) $\Gamma(f(x)^n) = \Gamma(g(x)^n)$,

for every integer $n \geq 1$.

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Saturday, December 7, 1985 — Examination B

B-1. Let k be the smallest positive integer with the following property:

> There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial
>
> $$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$
>
> has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

B-2. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

B-3. Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

B-4. Let C be the unit circle $x^2 + y^2 = 1$. A point p chosen randomly on the circumference C and another point q chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

B-5. Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\,dt$. You may assume that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

B-6. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.