FORTY-NINTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 3, 1988 — Examination A

A-1. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.

A-2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = \exp(x^2)$, determine, with proof, whether there exists an open interval (a, b) and a non-zero function g defined on (a, b) such that the wrong product rule is true for x in (a, b).

A-3. Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges.

A-4. (a) If every point on the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

(b) What if "three" is replaced by "nine"?

Justify your answers.

A-5. Prove that there exists a *unique* function f from the set $\mathbf{R}^+$ of positive real numbers to $\mathbf{R}^+$ such that

$$f(f(x)) = 6x - f(x) \quad \text{and} \quad f(x) > 0 \quad \text{for all} \quad x > 0.$$

A-6. If a linear transformation $\mathbf{A}$ on an n-dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that $\mathbf{A}$ is a scalar multiple of the identity? Prove your answer.

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Saturday, December 3, 1988 Examination B

B-1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2,3,4,\ldots\}$. Show that every composite is expressible as $xy+xz+yz+1$, with x, y, and z positive integers.

B-2. Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

B-3. For every n in the set $\mathbf{Z}^+ = \{1,2,\ldots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c+d=n$. Find, with proof, the smallest positive real number g with $r_n \leq g$ for all n in $\mathbf{Z}^+$.

B-4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

B-5. For positive integers n, let $\mathbf{M}_n$ be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of $\mathbf{M}_n$. (According to one definition the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$\mathbf{M}_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

B-6. Prove that there exist an infinite number of ordered pairs (a,b) of integers such that for every positive integer t the number $at+b$ is a triangular number if and only if t is a triangular number.

(The triangular numbers are the $t(n) = n(n+1)/2$ with n in $\{0,1,2,\ldots\}$.)