FIFTY-THIRD ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 5, 1992 Examination A

A-1. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

(i) $f(f(n)) = n$, for all integers n;

(ii) $f(f(n+2)+2) = n$ for all integers n;

(iii) $f(0) = 1$.

A-2. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 C(-y-1)\left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992}\right) dy.$$

A-3. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2+y^2)^m = (xy)^n$.

A-4. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0), k = 1, 2, 3, \ldots$.

A-5. For each positive integer n, let

$$a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+m+2j}, \quad \text{for } 0 \le j \le m-1.$$

A-6. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

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Saturday, December 7?, 1992 Examination B

B-1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

B-2. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = a!/(b!(a-b)!)$ for $0 \leq b \leq a$, and $\binom{a}{b=0}$ otherwise.)

B-3. For any pair (x,y) of real numbers, a sequence $(a_n(x,y))_{n\geq 0}$ is defined as follows:

$$a_0(x,y) = x,$$
$$a_{n+1}(x,y) = \frac{(a_n(x,y))^2 + y^2}{2}, \qquad \text{for } n \geq 0.$$

Find the area of the region $\{(x,y) | (a_n(x,y))_{n\geq 0} \text{ converges}\}$.

B-4. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3-x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

B-5. Let D_n denote the value of the $(n-1)\times(n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$$

Is the set $\{D_n/n!\}_{n\geq 2}$ bounded?

B-6. Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

(i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;

(ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;

(iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;

(iv) if $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.