A-1. Suppose that a sequence \(a_1, a_2, a_3, \ldots\) satisfies \(0 < a_n \leq a_{2n} + a_{2n+1}\) for all \(n \geq 1\). Prove that the series \(\sum_{n=1}^{\infty} a_n\) diverges.

A-2. Let \(A\) be the area of the region in the first quadrant bounded by the line \(y = \frac{1}{2}x\), the \(x\)-axis, and the ellipse \(\frac{1}{9}x^2 + y^2 = 1\). Find the positive number \(m\) such that \(A\) is equal to the area of the region in the first quadrant bounded by the line \(y = mx\), the \(y\)-axis, and the ellipse \(\frac{1}{9}x^2 + y^2 = 1\).

A-3. Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance \(2 - \sqrt{2}\) apart.

A-4. Let \(A\) and \(B\) be \(2 \times 2\) matrices with integer entries such that \(A, A+B, A+2B, A+3B, A+4B\) are all invertible matrices whose inverses have integer entries. Show that \(A+5B\) is invertible and that its inverse has integer entries.

A-5. Let \((r_n)\) be a sequence of positive real numbers such that \(\lim_{n \to \infty} r_n = 0\). Let \(S\) be the set of numbers representable as a sum
\[r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}},\]
with \(i_1 < i_2 < \cdots < i_{1994}\). Show that every nonempty interval \((a, b)\) contains a nonempty subinterval \((c, d)\) that does not intersect \(S\).

A-6. Let \(f_1, f_2, \ldots, f_{10}\) be bijections of the set of integers such that for each integer \(n\), there is some composition \(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}\) of these functions (allowing repetitions) which maps 0 to \(n\). Consider the set of 1024 functions
\[\mathcal{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}\},\]
e; = 0 or 1 for \(1 \leq i \leq 10\). \((f_i^0\) is the identity function and \(f_i^1 = f_i\).) Show that if \(A\) is any nonempty finite set of integers, then at most 512 of the functions in \(\mathcal{F}\) map \(A\) to itself.
B-1. Find all positive integers that are within 250 of exactly 15 perfect squares.

B-2. For which real numbers $c$ is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

B-3. Find the set of all real numbers $k$ with the following property: For any positive, differentiable function $f$ that satisfies $f'(x) > f(x)$ for all $x$, there is some number $N$ such that $f(x) > e^{kx}$ for all $x > N$.

B-4. For $n \geq 1$, let $d_n$ be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

Show that $\lim_{n \to \infty} d_n = \infty$.

B-5. For any real number $\alpha$, define the function $f_\alpha(x) = [\alpha x]$. Let $n$ be a positive integer. Show that there exists an $\alpha$ such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_\alpha^k(n^2).$$

B-6. For any integer $a$, set

$$n_a = 101a - 100 \cdot 2^a.$$  

Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a, b\} = \{c, d\}$. 