FIFTY-NINTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 5, 1998 Examination A

A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not of the position, of s.

A3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

A4. Let $A_1 = 0$ and $A_2 = 1$. For $n \geq 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $a_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

A5. Let $\mathcal{F}$ be a finite collection of open discs in $\mathbf{R}^2$ whose union contains a set $E \subseteq \mathbf{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$\cup_{j=1}^{n} 3D_j \supseteq E.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

A6. Let A, B, C denote distinct points with integer coordinates in $\mathbf{R}^2$. Prove that if

$$\big(|AB| + |BC|\big)^2 < 8 \cdot \big[ABC\big] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

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B1. Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x>0$.

B2. Given a point (a,b) with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at (a,b), one on the x-axis, and one on the line $y=x$. You may assume that a triangle of minimum perimeter exists.

B3. Let H be the unit hemisphere $\{(x,y,z): x^2+y^2+z^2=1, z\geq 0\}$, C the unite circle $\{(x,y,0: x^2+y^2=1\}$, and P a regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A\sin\alpha + B\cos\beta$, where A, B, α, β are real numbers.

B4. Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1}(-1)^{\lfloor i/m\rfloor+\lfloor i/n\rfloor}=0.$$

B5. Let N be the positive integer with 1998 digits, all of them 1; tht is,

$$N=1111\cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

B6. Prove that, for any integers a,b,c, there exists a positive integer n such that $\sqrt{n^3+an^2+bn+c}$ is not an integer.