A1. Show that every positive integer is a sum of one or more numbers of the form $2^r3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

A2. Let $S = \{(a, b) \mid a = 1, 2, \ldots, n, \ b = 1, 2, 3\}$. A rook tour of $S$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (i) $p_i \in S$, (ii) $p_i$ and $p_{i+1}$ are a unit distance apart, for $1 \leq i < 3n$, (iii) for each $p \in S$ there is a unique $i$ such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$? (An example of such a rook tour for $n = 5$ is depicted below.)

A3. Let $p(z)$ be a polynomial of degree $n$, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

A4. Let $H$ be an $n \times n$ matrix all of whose entries are $\pm 1$ and whose rows are mutually orthogonal. Suppose $H$ has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

A5. Evaluate

$$\int_0^1 \frac{\log(x+1)}{x^2 + 1} \, dx.$$ 

A6. Let $n$ be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are $n$ randomly, independently and uniformly, chosen points on a circle. Consider the convex $n$-gon whose vertices are the $P_i$. What is the probability that at least one of the vertex angles of this polygon is acute?
B1. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to $\nu$.)

B2. Find all positive integers $n, k_1, \ldots, k_n$ such that

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$ 

B3. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number $a$ such that

$$f'(\frac{a}{x}) = \frac{x}{f(x)}$$ 

for all $x > 0$.

B4. For positive integers $m$ and $n$, let $f(m, n)$ denote the number of $n$-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

B5. Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$(a) \quad \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \ldots, x_n) = 0 \quad \text{(identically)}$$ 

and that

$$(b) \quad x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$ 

Show that $P = 0$ identically.

B6. Let $S_n$ denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if $\pi$ is an even permutation and $\sigma(\pi) = -1$ if $\pi$ is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of $\pi$. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$