# SEVENTIETH ANNUAL
# WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 5, 2009 Examination A

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**A1.** Let $f$ be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points $P$ in the plane?

**A2.** Functions $f, g, h$ are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh}, \quad f(0) = 1,$$
$$g' = fg^2h + \frac{4}{fh}, \quad g(0) = 1,$$
$$h' = 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1.$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

**A3.** Let $d_n$ be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

**A4.** Let $S$ be a set of rational numbers such that

(a) $0 \in S$;

(b) If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$; and

(c) If $x \in S$ and $x \notin \{0, 1\}$, then $1/(x(x-1)) \in S$.

Must $S$ contain all rational numbers?

**A5.** Is there a finite abelian group $G$ such that the product of the orders of all its elements is $2^{2009}$?

**A6.** Let $f : [0,1]^2 \to \mathbf{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0,y)\,dy$, $b = \int_0^1 f(1,y)\,dy$, $c = \int_0^1 f(x,0)\,dx$, $d = \int_0^1 f(x,1)\,dx$. Prove or disprove: There must be a point $(x_0, y_0)$ in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$

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Saturday, December 5, 2009 Examination B

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**B1.** Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

**B2.** A game involves jumping to the right on the real number line. If $a$ and $b$ are real numbers and $b > a$, the cost of jumping from $a$ to $b$ is $b^3 - ab^2$. For what real numbers $c$ can one travel from 0 to 1 in a finite number of jumps with total cost exactly $c$?

**B3.** Call a subset $S$ of $\{1, 2, \ldots, n\}$ *mediocre* if it has the following property: Whenever $a$ and $b$ are elements of $S$ whose average is an integer, that average is also an element of $S$. Let $A(n)$ be the number of mediocre subests of $\{1, 2, \ldots, n\}$. [For instance, every subset of $\{1, 2, 3\}$ except $\{1, 3\}$ is mediocre, so $A(3) = 7$.] Find all positive integers $n$ such that $A(n + 2) - 2A(n + 1) + A(n) = 1$.

**B4.** Say that a polynomial with real coefficients in two variables, $x, y$, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space $V$ over $\mathbf{R}$. Find the dimension of $V$.

**B5.** Let $f : (1, \infty) \to \mathbf{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2((f(x))^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

**B6.** Prove that for every positive integer $n$, there is a sequence of integers $a_0, a_1, \ldots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after $a_0$ is either an earlier term plus $2^k$ for some nonnegative integer $k$, or of the form $b \bmod c$ for some earlier positive terms $b$ and $c$. [Here $b \bmod c$ denotes the remainder when $b$ is divided by $c$, so $0 \leq (b \bmod c) < c$.]