

**SEVENTY-FIRST ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION**

Saturday, December 4, 2010

Examination A

A1. Given a positive integer n , what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

A2. Find all differentiable functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n .

A3. Suppose that the function $h : \mathbf{R}^2 \rightarrow \mathbf{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a \frac{\partial h}{\partial x}(x, y) + b \frac{\partial h}{\partial y}(x, y)$$

for some constants a, b . Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbf{R}^2$, then h is identically zero.

A4. Prove that for each positive integer n , the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

A5. Let G be a group, with operation $*$. Suppose that

- (i) G is a subset of \mathbf{R}^3 (but $*$ need not be related to addition of vectors);
- (ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (or both), where \times is the usual cross product in \mathbf{R}^3 .

Prove that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for all $\mathbf{a}, \mathbf{b} \in G$.

A6. Let $f : [0, \infty) \rightarrow \mathbf{R}$ be a strictly decreasing continuous function such that $\lim_{x \rightarrow \infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} dx$ diverges.

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Examination B

B1. Is there an infinite sequence of real numbers a_1, a_2, a_3, \dots such that

$$a_1^m + a_2^m + a_3^m + \dots = m$$

for every positive integer m ?

B2. Given that A , B , and C are noncollinear points in the plane with integer coordinates such that the distances AB , AC , and BC are integers, what is the smallest possible value of AB ?

B3. There are 2010 boxes labeled $B_1, B_2, \dots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n . You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

B4. Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

B5. Is there a strictly increasing function $f : \mathbf{R} \rightarrow \mathbf{R}$ such that $f'(x) = f(f(x))$ for all x ?

B6. Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k , let $A^{[k]}$ be the matrix obtained by raising each entry to the k^{th} power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \dots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.