Day 05 – The Binomial Theorem (cont.)

Last time

**Theorem 1** (Binomial Theorem). For all non-negative integers \( n \),

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

**Corollary 1.1.** The following are direct consequences of Binomial Theorem.

1. For all positive integers \( n \), \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).
2. For all non-negative integers \( n \), \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).

**Theorem 2** (Pascal’s recursion). For all non-negative integers \( n, k \),

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

Combinatorial proof:

- **RHS** = \( 2^n = \text{number of binary strings of length } n \).
- **LHS** = \( \text{counting binary strings based on density (# 1-bits)} \).

\[
\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n} = \text{total # of bin. strings of length } n.
\]
Pascal's recursion: For all non-negative integers \( n, k \),

\[
\binom{n}{k} = \binom{n}{k+1} + \binom{n+1}{k+1}
\]

Combinatorial Proof:

\[
\text{RHS} = \binom{n+1}{k+1} = \# \{ (y, z_1, \ldots, z_{k+1}) : y \in [n+1], z_1, \ldots, z_{k+1} \in [n] \}
\]

\[
\text{If } (n+1) \text{ is in the subset.}
\]

\[
\text{Choose } (k+1) \text{- elements from } [n]
\]

\[
\binom{n}{k} + \binom{n}{k+1}
\]

\[
. \quad n = 10, \quad k = 4
\]

\[
(10) + (\binom{10}{4}) = (10 + 1) = (11)
\]

\[
. \quad n = 163, \quad k = 78
\]

\[
(163) + (163) = (169)
\]
Theorem 3 ("Hockey stick property"). For all non-negative integers \( n \geq k \),

\[
\sum_{i=k}^{n} \binom{i}{k} = \binom{k}{k} + \cdots + \binom{n-1}{k-1} + \binom{n}{k} = \binom{n+1}{k+1}
\]

Proof: \( \binom{n+1}{k+1} = \# \text{ \((k+1)\)-element subset of } [n+1] \)

Classify by the maximum entry.

\[
\begin{align*}
\text{max}(S) & = k + 1 \\
\text{max}(S) & = k + 2 \\
& \quad \vdots \\
\text{max}(S) & = n + 1
\end{align*}
\]

\( \binom{k}{k} \) way to \( \binom{k+1}{k} \) ways to \( \binom{k+2}{k} \) ways to \( \binom{k+i-1}{k} \) ways to \( \binom{n}{k} \)

got \( k \) elements \( \text{from } [k] \)

got \( k \) elements \( \text{from } [k+1] \)

3 \( \text{choose } k \text{ elements } \{1, 2, \ldots, k+i-1\} = [k+i-1] \)

\[
\begin{align*}
\binom{10}{10} + \binom{11}{10} + \binom{12}{10} + \cdots + \binom{17}{10} &= \binom{18}{11} \\
\text{RHS} &= \binom{n+1}{k+1}
\end{align*}
\]
Theorem 4. For all non-negative integers \( n \),

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n.
\]

Note: this result implies that \( \sum_{k=0}^{n} \binom{n}{k} \) is an integer which is not obvious!

*Algebraic Proof:*

\[
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]

Let \( y = 1 \):

\[
(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k
\]

Take \( \frac{d}{dx} \):

\[
\frac{d}{dx} x^{n-1} \quad \frac{d}{dx} \left( \sum_{k=0}^{n} \binom{n}{k} x^k \right) = \sum_{k=0}^{n} \frac{d}{dx} \left( \binom{n}{k} x^k \right)
\]

\[
\Rightarrow \quad n (x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}
\]

\[
\Rightarrow \quad n \cdot 2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}
\]
For all non-negative integers $n$, \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).

Combinatorial Proof:

Suppose we have $n$ players in a draft, and we want to draft a team then assign a captain. How many ways can we do this?

**RHS:** Choose someone to be a captain: \( \binom{n}{1} \)
- For each of the remaining $n-1$ players, \( n-1 \) ways
- 0 \( \rightarrow \) not picked
- 1 \( \rightarrow \) picked
- \( n-1 \) \#bin. stry of length $n-1 = \binom{n-1}{2}$

**LHS:** Fix $k$ the number of players I want to get
- For each $k$, need to choose $k$ players to be on my team: \( \binom{n}{k} \)
- Then I will assign 1 of these $k$ picks to be my captain: \( k \) ways

So we must have $\binom{n}{k}$ ways.

Note: When $k=0$, we have \( \binom{n}{0} = 1 \) way to pick zero player but then no one can be a captain, so \( \binom{n}{k} = 0 \).
Theorem 5 (Vandermonde's Convolution). For all positive integers $n, m, k,$

\[
\binom{n + m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}
\]

Proof:

\[
\binom{n+m}{k} = \# \text{ of } k\text{-element subsets from } \{1, 2, \ldots, m, m+1, m+2, \ldots, m+n\}.
\]
Definition 1. A sequence \( a_0, a_1, a_2, \ldots, a_n \) is said to be unimodal if there exists an index \( p \) such that \( a_0 \leq a_1 \leq \cdots \leq a_p \) and \( a_p \geq a_{p+1} \geq \cdots \geq a_n \).

Here, \( a_p \) is called the peak element and \( p \) is the peak index.

Theorem 6. The sequence \( \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{\frac{n}{2}}, \binom{n}{n} \) is unimodal with peak index \( n/2 \). That is,

\[
\binom{n}{k} \leq \binom{n}{k+1} \quad \text{for} \quad 0 \leq k < \frac{n}{2} \quad \text{and} \\
\binom{n}{k} \geq \binom{n}{k+1} \quad \text{for} \quad \frac{n}{2} \leq k \leq n.
\]