Day 07 – Compositions and Set Partitions

**Definition 1.** A sequence \( \alpha = (a_1, a_2, \ldots, a_k) \) such that \( a_i \geq 0 \) for all \( i \in [k] \), and \( a_1 + \ldots + a_k = n \) is called a **weak composition** of \( n \).

If \( a_1 + \ldots + a_k = n \) where \( a_i > 0 \) for all \( i \in [k] \) then \( \alpha = (a_1, a_2, \ldots, a_k) \) is called a **strong composition** of \( n \). In this class, “composition” means strong composition.

Here, \( k \) is the number of parts of \( \alpha \).

**Example.** Find all compositions of 4.

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**Theorem 1.** For all positive integers \( n \) and \( k \), the number of **weak compositions** of \( n \) into \( k \) parts is

\[
\binom{n + k - 1}{k - 1} = \binom{n + k - 1}{n - 1}.
\]

**Proof.**
Theorem 2. For all positive integers \( n \) and \( k \), the number of compositions of \( n \) into \( k \) parts is \( \binom{n-1}{k-1} \).

Proof.
Definition 2. A partition of a set $S$ is a collection of non-empty subsets/blocks such that every element in $S$ belongs to exactly one of the subsets/blocks.


Definition 3. 

- The number of set partitions of $[n]$ is given by $B(n)$, where $B(n)$ is the Bell number.
- The number of set partitions of $[n]$ with $k$ blocks is $S(n,k)$ - the Stirling number of the second kind.
- There is no closed formula for $B(n)$ and $S(n,k)$

Theorem 3. For any $n \leq 1$, the Bell numbers and Stirling numbers of the second kind satisfy the following properties:

- $B(n) = \sum_{k=1}^{n} S(n,k)$
- $S(n,k) = 0$ if $k > n$ or $k \leq 0$
- $S(n,1) = S(n,n) = 1$
- $S(n, n-1) = \binom{n}{2}$
Theorem 4. For $1 \leq k \leq n$, the Stirling numbers of the second kind satisfy the recursion

$$S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k)$$

Proof.
Function counting

How many functions \( f : [n] \to [k] \) are there?

How many \textbf{injection} \( f : [n] \to [k] \) are there?
How many surjection $f : [n] \rightarrow [k]$ are there?