Day 07 – Compositions and Set Partitions

**Definition 1.** A sequence \( \alpha = (a_1, a_2, \ldots, a_k) \) such that \( a_i \geq 0 \) for all \( i \in [k] \), and \( a_1 + \cdots + a_k = n \) is called a strong composition of \( n \).

If \( a_1 + \cdots + a_k = n \) where \( a_i > 0 \) for all \( i \in [k] \) then \( \alpha = (a_1, a_2, \ldots, a_k) \) is called a weak composition of \( n \). In this class, “composition” means strong composition.

Here, \( k \) is the number of parts of \( \alpha \).

**Example.** Find all compositions of 4.

\[
\begin{align*}
q &= 9 & 4 &= 4+1+2 & 4 &= 4+1+1+1 \\
4 &= 1+3 & = 1+2+1 & = 2+1+1 \\
\quad &= 2+2 & \quad &= 3+1
\end{align*}
\]

**Theorem 1.** For all positive integers \( n \) and \( k \), the number of weak compositions of \( n \) into \( k \) parts is

\[
\binom{n+k-1}{k-1} = \binom{n+k-1}{n}
\]

**Proof.**

Take a weak composition \( q \) of \( n \) into \( k \) parts

\[
(q_1, q_2, \ldots, q_k)
\]

where \( q_1 + q_2 + \cdots + q_k = n \).

\[
q_i \geq 0
\]
Theorem 2. For all positive integers $n$ and $k$, the number of compositions of $n$ into $k$ parts is \( \binom{n-1}{k-1} \).

Proof. \[
\begin{align*}
\sum a_1 + a_2 + \ldots + a_k &= n \\
a_i &> 1 \\
\text{use } n \text{ stars, } k-1 \text{ bars}
\end{align*}
\]

ex: $n=5$, $k=5$

\[
* + * + * + + + * + +
\]

Cannot place 2 bars next to one another
So we can only place bars in the red spots
one spot can have at most one bar.
Here, choose 4 spots out of 8: \( \binom{8}{4} \)

In general, \( \binom{n-1}{k-1} \)

\( \# \text{ of gaps in between the } *'s \)
\( \# \text{ of bars} \)

Earlier, \( \# \text{ composition of } n \) into

1 part: \( 1 = \binom{3}{0} = \binom{4-1}{1-1} \)

2 parts: \( \binom{3}{1} = 3 \)

3 parts: \( \binom{3}{2} = 3 \)

4 parts: \( \binom{3}{3} = 1 \)

Total \( \# \text{ of compositions} \)

\[
\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}
\]
Definition 2. A partition of a set $S$ is a collection of non-empty subsets/blocks such that every element in $S$ belongs to exactly one of the subsets/blocks.

Example. Find all set partitions of $[4] = \{1, 2, 3, 4\}$

1 block: $\{1, 2, 3, 4\}$

4 blocks: $\{1\}, \{2, 3, 4\}$, $\{1\}, \{2, 3, 4\}$

2 blocks: $\{1\}, \{2, 3, 4\}$, $\{1\}, \{2, 3, 4\}$

3 blocks: 2 singletons and 1 double

Definition 3.

- The number of set partitions of $[n]$ is given by $B(n)$, where $B(n)$ is the Bell number.

- The number of set partitions of $[n]$ with $k$ blocks is $S(n, k)$, the Stirling number of the second kind.

- There is no closed formula for $B(n)$ and $S(n, k)$.

Theorem 3. For any $n \geq 1$, the Bell numbers and Stirling numbers of the second kind satisfy the following properties:

- $B(n) = \sum_{k=1}^{n} S(n, k)$

- $S(n, k) = 0$ if $k > n$ or $k \leq 0$

- $S(n, 1) = S(n, n) = 1$

- $S(n, n-1) = \binom{n}{2}$

- n-elt and $n-1$ blocks then there's 1 double

- n-2 singletons

- choose 2 elements (out of $\binom{n}{2}$) to put in the double.
Theorem 4. For $1 \leq k \leq n$, the Stirling numbers of the second kind satisfy the recursion

$$S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k)$$

Proof.

$$S(n, k) = \# \text{ set partitions of } [n] \text{ into } k \text{-blocks.}$$

$$= \# \text{ of set partitions of } [n] \text{ into } k \text{-blocks} + \# \text{ set partitions of } [n] \text{ into } k \text{-block where } n \text{ is not in a singleton}$$

$$\downarrow$$

$$= \# \text{ of set partitions of } [n-1] \text{ into } k-1 \text{ blocks} + k \times \# \text{ of set partitions of } [n-1] \text{ into } k \text{ blocks}$$

$$= S(n-1, k-1) + k \cdot S(n-1, k)$$
* Make sure to read this for the exam!

Function counting

How many functions \( f : [n] \to [k] \) are there?

\[
\begin{align*}
  \text{k choices for } f(1) & \quad \text{k choices for } f(2) \quad \cdots \quad \text{k choices for } f(n) \\
  \quad \vdots & \quad \vdots \\
  \text{k choices for } f(n) & \quad = \quad \text{Total: } \frac{k!}{(k-n)!} \\
\end{align*}
\]

From \([k]\).

How many injection \( f : [n] \to [k] \) are there?

\[
\begin{align*}
  \text{k choices for } f(1) & \quad \text{k-1 choices for } f(2) \quad \cdots \quad \text{k-n+1 choices for } f(n) \\
  \quad \vdots & \quad \vdots \\
  \text{k choices for } f(n) & \quad = \quad \text{Total: } \frac{k!}{(k-n)!} \quad \text{Cif } k \geq n \\
\end{align*}
\]

So total: \( k \cdot (k-1) \cdot (k-2) \cdots (k-n+1) \)

Remark: If \( f : [n] \to [k] \) is an injection

then \( n = |[n]| \leq |[k]| = k \)

If \( n > k \) then there is no injection

\( f : [n] \to [k] \).
* I’ll do this on Wednesday. This will be on Midterm 2 and Final.

How many surjection \( f : [n] \to [k] \) are there?