Day 10 – Cycles in Permutations

Two exercises from HW3

Problem 1. Build the table for the Stirling number of the second kind \( S(n, k) \) for \( 1 \leq k \leq n \leq 6 \), and use your result to find the first six Bell numbers \( B(n) \) for \( 1 \leq n \leq 6 \).
Problem 2. List all integer partitions of 6 and decide which pair of partitions are conjugate of one another.
Cycles in permutations

A permutation of \([n]\) is a bijective function \(p : [n] \rightarrow [n]\). We denote \(S_n\) to be the set of all permutations of \([n]\).

There are three ways to write permutations

1. Function notation:

2. One-line notation:

3. Cycle notation:

In the canonical cycle form for permutations,

- Each cycle will be written with its largest element first, and
- The cycles will be arranged in increasing order of their first/largest element.

Fact: Any permutation can be written as a product of disjoint cycles.
Permutations can be multiplied/composed.

**Example.** In one-line notation, let \( \pi = 321564 \) and \( \sigma = 516234 \) be two permutations in \( S_6 \). Find \( \pi \cdot \sigma \).

Find \( \pi \cdot \sigma \) using the cycle notation.

Find \( \pi^2 = \pi \cdot \pi \).
Example. How many permutations of $S_9$ have three 2-cycles and one 3-cycle?
Theorem 1. Let \((a_1, a_2, \ldots, a_n)\) be non-negative integers such that

\[
1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n = n.
\]

Then the number of permutations \(\pi \in S_n\) with \(a_1\) 1-cycles, \(a_2\) 2-cycles, \ldots, and \(a_n\) \(n\)-cycles is given by

\[
\frac{n!}{1^{a_1}2^{a_2} \cdots n^{a_n} \cdot a_1!a_2! \cdots a_n!}
\]

Definition 1. The cycle-type of a permutation \(\pi \in S_n\) is \((a_1, a_2, \ldots, a_n)\) where \(a_i\) is the number of \(i\)-cycles in \(\pi\)
Stirling numbers of the first kind

Definition 2. The signless Stirling numbers of the first kind, $c(n,k)$, is the number of permutations of $S_n$ with $k$ cycles.

The (signed) Stirling numbers of the first kind, $s(n,k)$, is given by

$$s(n,k) = (-1)^{n-k}c(n,k).$$

Example. Find $c(4,3)$ and $s(4,3)$.

Theorem 2. The signless Stirling numbers of the first kind satisfy the following properties

1. $c(0,0) = 1$ and $c(n,0) = 0$ if $n > 0$.

2. Let $n$ be a fixed positive integer, then

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1).$$

3. For any $0 < k \leq n$, $c(n,k)$ satisfy the recursion

$$c(n,k) = c(n-1,k-1) + (n-1) \cdot c(n-1,k).$$