Day 11 – Stirling and Lah numbers

Last time

**Theorem 1.** Let \( (a_1, a_2, \ldots, a_n) \) be non-negative integers such that

\[
1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n = n.
\]

The **cycle-type** of a permutation \( \pi \in S_n \) is \( (a_1, a_2, \ldots, a_n) \) where \( a_i \) is the number of \( i \)-cycles in \( \pi \).

The number of permutations \( \pi \in S_n \) with cycle-type \( (a_1, a_2, \ldots, a_n) \)

\[
\frac{n!}{1^{a_1} 2^{a_2} \cdots n^{a_n} \cdot a_1! a_2! \cdots a_n!}
\]

**Definition 1.** The **signed Stirling numbers of the first kind**, \( c(n, k) \), is the number of permutations of \( S_n \) with \( k \) cycles.

The (signed) Stirling numbers of the first kind, \( s(n, k) \), is given by

\[
s(n, k) = (-1)^{n-k} c(n, k).
\]

**Theorem 2.** The signed Stirling numbers of the first kind satisfy the following properties

1. \( c(0, 0) = 1 \) and \( c(n, 0) = 0 \) if \( n > 0 \).

2. For any \( 0 < k \leq n \), \( c(n, k) \) satisfy the recursion

\[
c(n, k) = c(n-1, k-1) + (n-1) \cdot c(n-1, k).
\]
3. Let \( n \) be a fixed positive integer, then

\[
\sum_{k=0}^{n} c(n, k)x^k = x(x+1)(x+2)\cdots(x+n).
\]

Proof. First notice that

\[
\sum_{k=0}^{n} c(n, k)x^k = \sum_{k=0}^{\infty} c(n, k)x^k = \sum_{k=0}^{\infty} c(n, k)x^k =: F_n(x)
\]

\[
F_n(x) = \sum_{k=0}^{\infty} c(n, k)x^k
\]

\[
= \sum_{k=0}^{\infty} \left[ c(n-1, k-1) + (n-1) \cdot c(n-1, k) \right] x^k.
\]

\[
= \sum_{k=0}^{\infty} c(n-1, k-1)x^k + (n-1) \cdot \sum_{k=0}^{\infty} c(n-1, k)x^k.
\]

\[
(*) \equiv x \sum_{k=0}^{\infty} c(n-1, k-1)x^{k-1} = x \sum_{k=0}^{\infty} c(n-1, k)x^k
\]

\[
\therefore (*) = (n-1) \sum_{k=0}^{\infty} c(n-1, k)x^k = (n-1) F_{n-1}(x)
\]

\[
(**) = (n-1) \sum_{k=0}^{\infty} c(n-1, k)x^k = (n-1) F_{n-1}(x)
\]
\[
F_n(x) = x \cdot F_{n-1}(x) + (n-1) \cdot F_{n-2}(x) \\
\text{Proof (cont.)} \Rightarrow (x+n-1) \cdot F_{n-1}(x)
\]

So,

\[
F_n(x) = (x+n-1) \cdot F_{n-1}(x)
\]

\[
= (x+n-1) \cdot (x+n-2) \cdot F_{n-2}(x)
\]

\[
= (x+n-1) \cdot (x+n-2) \cdot (x+n-3) \cdot F_{n-3}(x)
\]

\[
\vdots
\]

\[
= (x+n-1) \cdot (x+n-2) \cdot \ldots \cdot (x+1) \cdot F_0(x)
\]

\[
F_0(x) = \sum_{k=0}^{\infty} c(0,k) x^k = c(0,0) x^0 = 1
\]

So,

\[
\sum_{k=0}^{n} c(n,k) x^k = \frac{x^n}{n+1}(x+1)(x+2) \ldots (x+n) \quad (***)
\]

Replace \(x\) by \((-x)\) in (***)

\[
\sum_{k=0}^{n} c(n,k)(-1)^k x^k = (-x)(-x+1)(-x+2) \ldots \ldots (-x+n-1)
\]

\[
= (-1)^n x(x-1)(x-2) \ldots (x-n+1)
\]

\[
\sum_{k=0}^{n} c(n,k)(-1)^k x^k = (x)_n \Rightarrow \sum_{k=0}^{n} c(n,k) x^k = (x)_n
\]

\[
\sum_{k=0}^{n} S(n,k) x^n = x^n
\]
\[ \sum_{k=0}^{n} S(n, k) \cdot (x)^{n-k} = x^n \]

**Relationship between \( s(n, k) \) and \( S(n, k) \)**

Consider the infinite matrices

\[ S_1 = [s(n, k)]_{n \geq k \geq 0} = \begin{bmatrix} s(0, 0) & s(1, 0) & s(2, 0) & s(3, 0) & \cdots \\ s(1, 1) & s(2, 1) & s(3, 1) & s(3, 2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]

\[ S_2 = [S(n, k)]_{n \geq k \geq 0} = \begin{bmatrix} S(0, 0) & S(1, 0) & S(2, 0) & S(3, 0) & \cdots \\ S(1, 1) & S(2, 1) & S(3, 1) & S(3, 2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]

Then \( S_1 S_2 = S_2 S_1 = I \)

\( \mathcal{M} = \{1, x, x^2, x^3, \ldots \} \) (monomial basis)

\( \mathcal{R} = \{1, x, (x)_2, (x)_3, \ldots \} \) (rising factorial)

\( \mathcal{L} = \{1, (x), (x)_2, (x)_3, \ldots \} \) (falling factorial basis)

\( \mathcal{T} = \{1, (x), (x)_2, (x)_3, \ldots \} \) (falling factorial basis)
Definition 2. The Lah number $L(n, k)$ is the number of ways to partition $[n]$ into $k$ non-empty, linearly ordered subsets.

Example.  
\[ L(3, 2) = \begin{pmatrix} 1/32 \\ 2/31 \\ 3/21 \end{pmatrix} \]

The Lah numbers satisfy the following properties:

1. $L(0, 0) = 1$ and $L(n, 0) = 0$ for all integers $n > 0$
2. $L(n, n) = 1$ and $L(n, 1) = n!$ (directly counted as the number of permutations of $n$ elements)
3. $x(x+1)(x+2) \cdots (x+n-1) = \sum_{k=0}^{n} L(n, k) \cdot (x-k)(x-k-1) \cdots (x-n+1)$
4. $L(n, k) = L(n-1, k-1) + (n+k-1) \cdot L(n-1, k)$

5. $L(n, k) = \frac{n!}{k!(n-k)!}$

Note: for cycle, inserting $n$ at the start or the end give us the same cycle.
\[ (1, 2, 3, 4) = (2, 3, 4, 1) \]
\[ (7, 1, 2, 3) = (1, 2, 3, 7) \]
Definition 3. Let \( n \) be a positive integer. Define

\[
\begin{align*}
\text{ODD}(n) & \text{ to be the set of permutations of } S_n \text{ with odd cycle lengths} \\
\text{EVEN}(n) & \text{ to be the set of permutations of } S_n \text{ with even cycle lengths}
\end{align*}
\]

Theorem 3. For all positive integers \( n \),

\[
|\text{ODD}(2n)| = |\text{EVEN}(2n)| = 1^2 \cdot 3^2 \cdot 5^2 \cdots (2n - 1)^2.
\]

Proof.

Set up a bijection \( f: \text{ODD}(2n) \rightarrow \text{EVEN}(2n) \).

Let \( \pi \in \text{ODD}(2n) \) whose cycle form is \( \pi = C_1 C_2 \cdots C_{2k} \).

- \( \pi \) must be in canonical cycle form.
- All \( C_i \)'s have odd length.

Define \( f(\pi) \) by taking the last element of each \( C_{2i-1} \) and put it at the end of \( C_i \).

\[
\pi = (4)(5,1,3)(7,2,4)(8) \rightarrow f(\pi) = (5,1,3,4)(7,2)(8,6)
\]

\( f(\pi) \) also in canonical cycle form.

**Question**: How do we reverse the mapping?

Find the pre-image of \( (5,1,3,4)(7,2)(8,6) \)?
Proof (cont).

Lemma 4 (Transition Lemma). Let \( p \in S_n \) be a permutation written in canonical cycle notation. Let \( g(p) \) be the permutation obtained by omitting the parentheses and reading the entries as a permutation in one-line notation. Then \( g \) is a bijection.

Example.