Day 14 & 15 – Ordinary Generating Functions

Generating Function

Let $t_n =$ the number of ways to tile a $2 \times n$ board using dominoes. Find a recursion formula for $t_n$. 
Can we find a closed formula for $t_n$, for any non-negative integer $n$?
Partial fraction: Write \( \frac{1}{f(x)} = \frac{A}{x - \alpha} + \frac{B}{x - \beta} \)
Definition 1. Let \((a_n)_{n \geq 0}\) be a sequence of real numbers. The ordinary generating function of \((a_n)_{n \geq 0}\) is defined by

\[
A(x) := \sum_{n \geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots
\]

Example. Let \(a_n = 3^n\), then the OGF of \((a_n)_{n \geq 0}\) is

Example. Let \(b_n = n\), then the OGF of \((b_n)_{n \geq 0}\) is

Remark: We treat \(\sum_{n \geq 0} a_n x^n\) as a formal power series, regardless of its radius of convergence.
Example. Let \( a_0 = 0 \) and \( a_{n+1} = a_n + 2^n \) for all \( n \geq 0 \). Find an explicit formula for \( a_n \) (no recurrence).

Method 1: Guess and Check

Method 2: Generating function
Operations on Generating Functions

Let \( A(x) = \sum_{n \geq 0} a_n x^n \) and \( B(x) = \sum_{n \geq 0} b_n x^n \). Then

- \( A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n \)
- \( A(x) \cdot B(x) = \sum_{n \geq 0} c_n x^n \) where \( c_n = \sum_{i,j \geq 0; \ i+j=n} a_i b_j \)

Combinatorial Interpretation for \( A(x) + B(x) \)

For \( n \geq 0 \), let

- \( a_n = \) number of ways to put an \( A \)-structure on an \( n \)-element set,
- \( b_n = \) number of ways to put an \( B \)-structure on an \( n \)-element set.

If \( A(x) = \sum_{n \geq 0} a_n x^n \) and \( B(x) = \sum_{n \geq 0} b_n x^n \) and \( A(x) + B(x) = \sum_{n \geq 0} c_n x^n \). Then \( c_n = \) number of ways to put an \( A \)-structure or an \( B \)-structure but not both, on an \( n \)-element set