Day 14 & 15 – Ordinary Generating Functions

Definition 1. Let \((a_n)_{n \geq 0}\) be a sequence of real numbers. The ordinary generating function of \((a_n)_{n \geq 0}\) is defined by

\[
A(x) := \sum_{n \geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots
\]

Remark: We treat \(\sum_{n \geq 0} a_n x^n\) as a formal power series, regardless of its radius of convergence.

Last time. We obtained the generating function

\[
A(x) = \sum_{n \geq 0} F_n x^n = \frac{1}{1 - x - x^2}
\]

where \(F_n\) is the \(n\)-th Fibonacci number.

To expand this RHS for \(0 \leq n \leq 10\), in Wolfram Alpha or Mathematica, type

\[
\gg\text{ Series}[1/(1 - x - x^2), \{x, 0, 10\}]
\]

This will give us the Taylor expansion of the RHS.

We also obtained a close formula for each \(F_n\) by doing partial fraction to the RHS and apply the formula of geometric series.

Example. Let \(a_0 = 0\) and \(a_{n+1} = a_n + 2^n\) for all \(n \geq 0\). Find an explicit formula for \(a_n\) (no recurrence).

Method 1: Guess and Check
Method 2: Generating function
Operations on Generating Functions

Let \( A(x) = \sum_{n \geq 0} a_n x^n \) and \( B(x) = \sum_{n \geq 0} b_n x^n \). Then

- \( A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n \)

- \( A(x) \cdot B(x) = \sum_{n \geq 0} c_n x^n \) where \( c_n = \sum_{i,j \geq 0; \ i+j=n} a_i b_j \)

Combinatorial Interpretation for \( A(x) + B(x) \)

For \( n \geq 0 \), let

- \( a_n \) = number of ways to put an \( A \)-structure on an \( n \)-element set,
- \( b_n \) = number of ways to put an \( B \)-structure on an \( n \)-element set.

If \( A(x) = \sum_{n \geq 0} a_n x^n \) and \( B(x) = \sum_{n \geq 0} b_n x^n \) and \( A(x) + B(x) = \sum_{n \geq 0} c_n x^n \). Then \( c_n \) = number of ways to put an \( A \)-structure or an \( B \)-structure but not both, on an \( n \)-element set.
Combinatorial Interpretation for $A(x) \cdot B(x)$

$$A(x) \cdot B(x) = \sum_{n \geq 0} c_n x^n$$

where $c_n = \sum_{i,j \geq 0; \ i+j=n} a_i b_j$

Then $c_n$ is the number of ways to break the line of $n$ objects at some position $i$ then place an $A$-structure on $\{1, \ldots, i\}$ and place a $B$-structure on $\{i+1, \ldots, n\}$.

**Example.** Suppose we have $n$ ice cream cones in a line. We divide the cones into two groups such that the first group contains the first $i$ cones, for $1 \leq i \leq n-1$ and the second group contains the remaining $n-i$ cones.

We then fill the first $i$ cones with either mint or chocolate, and fill the remaining $n-i$ cones with strawberry, vanilla, or coffee.

Let $c_n$ be the number of ways to do this. Find a closed formula for $c_n$. 
Example. There are $n$ soldiers lining up. We want to break the line at two places and assign a leader to each team. Let $c_n =$ number of ways to do this. Find $C(x) = \sum_{n \geq 0} c_n x^n$. 
Theorem 1. Let $a_0 = 0$ and let $a_n$ be the number of ways to place an $A$-structure on a $n$-element set ($n > 0$). Define $A(x) := \sum_{n \geq 0} a_n x^n$. Then
\[
\frac{1}{1 - A(x)} = 1 + A(x) + A^2(x) + \cdots = \sum_{n \geq 0} b_n x^n
\]
where $b_n =$ number of ways to split the line $1, 2, \ldots, n$ into any number of sub-lines and place an $A$-structure on each sub-line.

Example. We have $n$ ice cream cups on a line. We break the line into any number of sub-lines. For each sub-line, we pick two cups and place a cherry on top. Let $c_n =$ the number of ways to do this. Find $C(x) = \sum_{n \geq 0} c_n x^n$. 