Day 17 – Generating Functions for Integer Partitions

Partition Identities

Recap:

• A sequence of positive integers $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ is called an (integer) partition of $n$ provided that $a_1 + \cdots + a_k = n$. Denote $\lambda \vdash n$.

• The partition number $p(n)$ is the total number of integer partitions for $n$.

• $p_k(n)$ is the number of partitions of $n$ into $k$ parts.

Partition may be represented by Ferrers diagram.

• We write $|\lambda| = n$ denote that there are $n$ boxes in the Ferrer diagram of $\lambda$.

• We write $\ell(\lambda) = k$ to denote that there are $k$ parts in $\lambda$. 
**Question:** Given integers $k \leq n$. How many partition $\lambda$ of $n$ whose parts lengths are no longer than $k$ (i.e. $\lambda_1 \leq k$) are there?

**Answer.** Let $p_{\leq k}(n)$ denote the number of partitions of $n$ whose parts lengths are no longer than $k$.

Consider

$$P_{\leq k}(x) = \sum_{n \geq 0} p_{\leq k}(n)x^n = \sum_{n \geq 0} (# \text{ of } \lambda \vdash n \text{ with } \lambda_1 \leq k) \cdot x^n.$$ 

Then

$$P_{\leq k}(x) = \prod_{i=1}^{k} \frac{1}{1-x^i} = \left( \frac{1}{1-x} \right) \left( \frac{1}{1-x^2} \right) \cdots \left( \frac{1}{1-x^k} \right)$$
We also have

$$
P_k(x) = \sum_{n \geq 0} p_k(n)x^n = x^k \cdot \sum_{n \geq 0} p_{\leq k}(x)x^n = \prod_{i=1}^{k} \frac{x^i}{1 - x^i}.
$$
Let $p_{\text{odd}}(n)$ be the number of partitions of $n$ into \textbf{parts of odd length}. Let $p_{\text{dist}}(n)$ be the number of partitions of $n$ into \textbf{parts with distinct lengths}.

What are $O(x) = \sum_{n \geq 0} p_{\text{odd}}(n)x^n$ and $D(x) = \sum_{n \geq 0} p_{\text{dist}}(n)x^n$?
What are $O(x) = \sum_{n \geq 0} p_{\text{odd}}(n)x^n$ and $D(x) = \sum_{n \geq 0} p_{\text{dist}}(n)x^n$? (cont.)
Example. Consider the sequence given by \( a_0 = 0 \) and

\[ a_{n+1} = (n + 1)a_n + 2(n + 1)! \]

for \( n \geq 0 \). Find the explicit formula for \( a_n \).